Unit 7

Thinking Things Through Thoroughly

CONTENTS

T4 Unit Introduction
T9 Lesson 1: What Can You Say for Sure?
T11 Lesson 2: Asking Good Questions
T14 Lesson 3: Repeating and Generalizing
T17 Student Reflections & Snapshot Check-in
T18 Lesson 4: Mapping It Out
T20 Lesson 5: Logic Games
T23 Lesson 6: Liars and Truth Tellers
T26 Student Reflections & Unit Assessment
T27 Exploration: Staircase Pattern
T29 Exploration: Arrangements in a Circle
T31 Game: Coin Challenge

RESOURCES

T32 Futoshiki Puzzle (Lesson 2)
T33 Mapping It Out (Lesson 4)
T34 Snapshot Check-in
T35 Snapshot Check-in Answer Key
T36 Unit Assessment
T38 Unit Assessment Answer Key

T40 MENTAL MATHEMATICS

Distributive property and distance
T41 Doubling with units digit greater than 4
T42 Doubling, challenges
T43 Halving even integers with odd tens digits
T44 Halving, challenges
T45 Distance to 100
T46 Distance to 10 with decimals
T47 Distance to 1 with fractions
T48 Distance to 10 with mixed numbers
T49 Distance to 30 with thirds
T50 Distance to 30.5
T51 Distance to 30.4 with decimal numbers
T52 Choose a review
Learning Goals

By the end of Unit 7, students should be able to:

• Build stamina for tackling challenging problems.
• Develop the habit of asking, “What can I do?” instead of “What am I supposed to do?” to gain entry points to problems.
• When given a mathematical context, determine what can be said for sure.
• Become producers of mathematical language and be able to pose problems that can be answered uniquely based on a set of given information.
• Learn to use the repeat and generalize strategy for creating algebraic equations.
• Build facility with organizing information, considering all cases, and persevering to a conclusion.

The problems in this unit are designed to stimulate mathematical thinking and to offer more practice solving problems, a skill that takes time and experience to develop. Problem solving requires a kind of thinking that cannot quite be captured in “steps.” Students must look for what is familiar and unfamiliar in the problem, look for entry points, poke around a bit, pay attention to hunches and intuitions, and assess whether their trials and experiments move them closer to an insight or a solution. Unit 7 invites students to consider what they can determine from a context and what possibilities there are before focusing on the answer requested. Students use tables and diagrams to organize information, and learn to “repeat and generalize” calculations to produce algebraic equations to describe problem contexts.

Word Problems

Students are consumers of the language of word problems in their tests and texts. In order to interpret that language well, they need to learn it, and students learn language best when they are interacting with it, producing it as well as interpreting it. The Transition to Algebra approach to word problems, therefore, gets students to produce the kind of language they will later have to decode on tests and in courses after algebra.

In Lessons 1 and 2 of Unit 7, students consider situations—essentially word problems without the questions—and ask themselves, “What can I say for sure?” and “What questions can I ask?” These more general questions require a deeper and more flexible analysis of the given information and make it harder to slip into the all-too-common mode of trying to find a way to combine the given quantities into an answer.
You can extend this idea into your teaching practice beyond *Transition to Algebra*: take a standard word problem and remove the question, leaving the rest of the problem intact. Present what remains and the following challenge:

- **What can be figured out from this information?** Students don’t have to do the figuring out; just say what can be figured out. Very often, more than one thing can be figured out.
- **What mathematical questions can be asked?** Often, many good questions can be asked. Again, students don’t have to answer the questions; just figure out what questions make sense in this context or what additional information could be sought.

This challenge frees students from wondering what they are “supposed to do” and encourages independent mathematical thinking. You may give students time to think on their own or invite them to write a response or two of their own, but *discussion is essential*. In discussion, students begin to see the variety of questions that can be asked, and this will increase their repertoire of ideas and appreciation for considering the whole problem—not just the numbers and how they are “supposed to” combine them.

It takes time to build up the confidence and approach that makes students good problem solvers, but any one dose of this activity should last *no more than 15 minutes* and generally less. Lessons 1 and 2 introduce students to this kind of thinking.

**What Can You Say for Sure about the Coins in My Hand?**

Hold up a closed hand and say: “I have exactly 9¢ in my hand. You might guess what coins I’m holding, but you can’t say for sure what they are. But there is a lot you can say for sure. For example . . . .” Then pause and wait for the class to respond with ideas.

In students’ first encounters with this kind of task, they are almost never especially forthcoming. The idea is unfamiliar, and they likely expect that there is some “right answer” that you are waiting for and they don’t know. Students tend to be afraid that their ideas are not worth saying, sound trivial, and so on. In fact, there is real learning for students in distinguishing what *is* trivial (You have 9¢.) from what *feels* trivial but isn’t (You could have nine pennies.). So part of your initial job as the teacher is to accept even what sounds trivial and help students *figure out* what is not trivial about it. It will often take fast thinking on your part to figure out what is not trivial about some answers!

There are many good responses possible for most of these problems. Mathematical thinking and communication about problems that have multiple correct answers can free students to consider what they can discover with the given information. Encourage all relevant responses, especially at first, no matter how “obvious” they might seem. Listen to, and maybe even record, all students’ mathematical ideas—right or wrong—and try not to be the first to respond to their statements, so that students get used to responding to each
What if . . .

What if students don’t know how to start? (Or they can start but say only one or two things and then seem unable to go on.)

Because the analysis and computational demand in this first situation—creating some image of 9¢—is so light, it is safe to say that silence is never an indication that students cannot think of anything to say. Almost always, the problem is self-censorship: they think of things but feel foolish saying them. “There must be some ‘right’ thing to say, something more the teacher wants, something I don’t get, something ‘smarter’ than what I thought of.” The following ideas often help students open up.

- **Time:** Leaving silence is often enough. Imagine the alphabet song in your head, and leave that amount of silence, without repeating the question, asking for volunteers, pulling for answers, or even encouraging responses. Students often find the quiet astonishing and a bit awkward, and they may find something to say just to fill the space. As long as their response addresses the question you posed, find a way to use it and show its relevance.

- **Permission to say “simple” things:** If time isn’t enough, suggest, “One thing you can say for sure is that I have no dollar bills in there.” Write down: “No dollar bills,” and ask, “What else could you say for sure?” This helps clarify that you are not waiting for some elusive, super-deep answers. Anything that is mathematically relevant and true is good enough.

- **Modeling:** Use this only as a last resort, because your goal is not to show how much you can answer! Give more time, and if necessary, another “trivial” response. But if students are still having a hard time, suggest that they picture what could be in your hand and describe it. By giving them one strategy—picturing the situation—you hope to get them started talking about it. And if this is not enough, move on. Let the class know that they’ll see more problems like this and they’ll find it easier over time. It is hard.

Other. Later, probe for underlying thinking both to elucidate ideas for others and to strengthen the reasoning of the speaker. Try to create an environment that encourages students to safely share their ideas and thinking.

Three reactions are common:

1. For most students, tasks like this are utterly unfamiliar, and the first time you do this you may get dead silence. Just wait in silence (count to 20 inside your head) without repeating the question or prompting; this gives space to think.

2. You may get one or two statements and then dead silence.

3. Over time (but not likely at first), questions like this will inspire a flurry of varied, relevant, and mathematically interesting statements. Here are just a few of the possible (correct) responses that could be given, to give you a sense of how much can be said even about this simple situation.
   - You could have 4 pennies and a nickel.
   - You could have 9 pennies.
   - You must have some pennies.
   - You don’t have a dime.
   - You cannot have any coin worth more than a nickel.
   - You must have at least 4 pennies.
   - You must have at least 5 coins.
   - You cannot have more than 9 coins.
   - You must have either 9 pennies or 4 pennies and a nickel.
   - You must have an odd number of coins.
   - You can’t share that equally with one other person.
   - Depending on what coins you have, you might be able to share what you have equally with two other people.

Acknowledge each correct student response with a clear “yes,” and write the response on the board. Below are various types of student responses and reasonable teacher follow-ups.

- **Responses that seem trivial,** such as “You have 9¢ in your hand” or “You don’t have any quarters.” Though these seem trivial, they are, of course, true. Acknowledge statements like these and write them on the board. Responses such as these should be taken just as seriously as a more interesting response—especially as students are first encountering this exercise—because all responses are, in a sense, obvious once you understand them thoroughly. Acknowledging all responses ensures that students don’t self-censor what seems obvious or trivial to them. (See “Perseverative responses” below.)

- **Assertions of a possibility,** such as “You could have 4 pennies and a nickel.” All assertions of a possibility, as long as they are correct, can be said for sure. Be careful not to push for a generalization (e.g. “You have
to have at least some pennies”). This steals the chance for students to discover the generalization themselves—perhaps after someone else says, “Or you could have 9 pennies.” Focus on soliciting the maximum number of mathematically correct statements so students realize how much can be said even about a situation that seems so straightforward.

- **Possibilities stated as facts**, such as “You have 4 pennies and a nickel.” Watch for possibilities that are stated as facts. This statement can’t be said for sure with the information that students have. Help them distinguish between a guess and a statement of fact by using language like, “You can’t say that for sure, but you can say that I could have 4 pennies and a nickel, so we’ll write that down.” You want to work against guessing, but acknowledge that guesses often contain good thinking. This is just like the step in introducing MysteryGrid puzzles where you taught students to record everything they figure out, but to write down only what they know for sure and to do so in a way that they will not mistake notes (written in small, light pencil marks) for an answer.

- **Questions**, such as “Is it 4 pennies and a nickel?” When students ask questions, try to get them to answer the questions themselves. “How would you answer that question?” “I can’t answer. I can’t see what’s in your hand.” “Right, you can’t know for sure, but your question shows that you do know something. Why did you ask ‘Is it 4 pennies and a nickel?’ instead of, say, ‘7 pennies and a dime’?” “Because it could be 4 pennies and a nickel.” “Right! So we can write that down for sure.” Students’ questions are often based on assumptions, and you want them to find and acknowledge the relevant knowledge that they do have. Often, students’ mistakes on mathematical tasks are rooted in having ignored something they know or having assumed something was a fact that was actually just a possibility.

- **Perseverative responses**: Sometimes students hit on an idea like “You can’t have any dimes” and then follow it with near clones like “You can’t have any quarters” and “You can’t have any dollar bills” and so on. Accept the first few repetitions and don’t instantly push for generalization. A repetition may occur as part of the process of coming to notice a generalization (as students will learn to utilize in Lesson 3). If the class provides several repetitive responses, you can point out the intelligence of it by acknowledging that they are making all these statements because they have noticed the general rule. You might say something like “Ah, right! You’re saying ‘You can’t have anything bigger than a nickel.’ Good! Let’s save me some space here and just write that.” It is because they have noticed the generalization that they can repeat other similar possibilities so quickly.
Mental Mathematics: Distributive Property and Distance

The Mental Mathematics in this unit builds on activities from previous units. Students practice doubling and halving (introduced and developed in Units 1 and 5) with inputs that continue to focus attention on the distributive property. These activities maintain and increase students’ sense of success. Likewise, several distance activities build on knowledge students have gained in Units 2, 3, and 6. Students also increase competence with rational numbers by extending to decimals and fractions the thinking they have applied to integers.

Explorations

In the Staircase Pattern Exploration, students investigate the structure of triangular numbers using drawings, and they develop a formula for the \(n\)th stage of the pattern they see. In Unit 8, students will revisit this pattern in its more familiar form and will discover again the algebraic expression for each stage in the pattern. The Arrangements in a Circle Exploration is intended to help students learn to reason based on simpler cases. Students explore the arrangement of numbers in a circle that are either “odd” or “even” and consider the number of odd-odd links, even-even links, odd-even links, and total links in circles of different numbers of even and odd numbers under various constraints. This supports important themes in Unit 7: considering possibilities and ordering elements.

Related Game: Coin Challenge

The Coin Challenge game has students regularly revisit the question “What can be said for sure?” as they consider clues about actual coins that you have in your hand and use logic to determine what you have. You will offer a statement such as “I have exactly 13 coins totaling 43¢ in my hand” and ask students to determine how many coins you have. Collect a set of 2 dimes, 5 nickels, and 18 pennies to select subsets from so that you can jingle them as students consider the problem.
Lesson 1: What Can You Say for Sure?

PURPOSE
Solving word problems requires deduction. In this lesson, students learn to derive what they can from given information, before seeing a formal question. This lesson focuses on one of the major themes of Unit 7: finding entry points for problems. Students learn to place emphasis on logic as they deduce what they can using tables to organize information and check for all possibilities. After determining what can be said, students encounter questions that use language from textbooks and tests and can be answered from the observations they’ve made.

Mental Mathematics Begin each day with five minutes of Mental Mathematics (pages T40–T52). The activities of this unit will pull together skills learned from previous Mental Mathematics activities and continue to push students’ ability to hold and manipulate mathematical information in their heads.

Launch: What Can You Say for Sure about the Coins in My Hand?
Hold up a closed hand and say: I have exactly 9¢ in my hand. You might guess what coins I’m holding, but you can’t say for sure what they are. But there is a lot you can say for sure. For example… Then pause and wait for the class to respond with ideas. For more about leading this discussion, refer to the Unit 7 Introduction in this Teaching Guide.

Student Problem Solving and Discussion
Have students work in the Student Worktext in small groups. Every student should do the Important Stuff PROBLEMS 1–3.

Encourage students to try Stuff to Make You Think and Tough Stuff problems. Be sure students stay focused on mathematical problems throughout the whole class period.

The goal is to ensure that students grasp the following key ideas:

- Effective, productive process is the new thing to learn. The focus, for now, is not on the answers. It is often difficult to make this argument convincingly, because correct answers do, of course, ultimately matter. But that’s not the focus now. Use the problems in this unit to help make your case.

- Many problems in mathematics (and in life) aren’t solved quickly.
Encourage students to revisit previous problems or to explore ideas of their own.

Lesson at a Glance
Preparation
- Prepare 4 pennies and 1 nickel for the Launch.
- Choose one problem from page T31 for today’s Coin Challenge game, and prepare the necessary coins.

Mental Mathematics (5 min)
Launch: What Can You Say for Sure about the Coins in My Hand? (10 min)
- Students consider possibilities based on knowing that you have 9¢ in your hand.

Student Problem Solving and Discussion (20 min)
- Students consider several contexts in which they determine what can be said for sure, organize the possibilities, and then answer several word-problem-style questions about each context.

Unit 7 Related Game: Coin Challenge (See page T31 and Student Worktext page 37.)

The problems in Unit 7 are designed to help students solve word problems that appear in mathematics classes and on standardized tests. However, these problems are designed to look different because the approach is different, and also so students don’t feel like these are problems they are already expected to know how to do.
• Students should focus on what they can do without worrying about whether it gets them to the answer right away. Again, the answer is actually a less important aspect of most of the problems in Unit 7. Don’t give answers away too quickly, and teach students how to record their attempts and build upon them.

Use some of these discussion prompts to engage students in discussing the idea of certainty more deeply:

» If you spend all of the money in your bank account, what can you say for sure? This question connects the idea of certainty to an idea many students face at some point. Students may think of ideas like the following: you’ll either have no money or negative money, you’ll need to put in more money to keep spending, you shouldn’t write a check, you might need a better-paying job, etc.

» In what other situations could you ask about what’s knowable for sure? This is another chance to get students generating their own mathematical ideas. They don’t have to answer the questions, just ask. This frees students to be mathematically creative. Other students can share their answers if they think they know something for sure. Record all ideas to inspire more questions. This prepares students to ask good questions, as they will practice more in Lesson 2.

» What types of things can we say for sure about problems like these in general? Listen for words like “could,” “must,” and “can’t” and guide students to distinguish between possibilities and certainties, perhaps by organizing responses. Students may identify different categorization schemes.

» If you know I have 7¢, and you notice that I can’t have any dimes, quarters, dollar bills, or higher denominations, what can you conclude about the coins I could have? This is an opportunity for students to begin generalizing from their repeated reasoning. The goal is to abstract the idea “You can’t have anything bigger than a nickel” by combining several observations: “no quarters,” “no dimes,” “no bills,” etc. Inspire students to consider the idea of generalizing by asking: Could I have a $20 bill? A $10 bill? $100? This prompts for the abstraction “You cannot have any bills at all.” You want students to seek general rules. In Lesson 3, Repeating and Generalizing, students will generalize repeated calculations to create algebraic notation that describes a situation.
Ask students to reflect on their learning:

**What are some things you’ve learned so far in this unit?**
**What questions do you still have?**

Assess student understanding of the ideas presented so far in the unit with the Snapshot Check-in on page T34. Use student performance on this assessment to guide students to select targeted Additional Practice problems from this or prior lessons as necessary.

**Student Reflections & Snapshot Check-in**

**So far in Unit 7, students have:**

- Considered what they can say *for sure* about a given context and organized the possibilities using tables and repeated calculations.
- Identified additional information that would be helpful in solving a problem and reflected on various potential directions of inquiry.
- Constructed equations by exploring a mathematical situation several times with different numbers to feel the rhythm of the process and then repeating that process with a variable and recording each step.

**Students have been developing the following Algebraic Habits of Mind:**

- **Puzzling and Persevering**—Students have learned to attend to the meaning of a problem’s context before diving into calculations. This supports focused and thoughtful problem solving, in contrast to rushing straight to combining numbers without first understanding what the problem is asking.
- **Using Tools Strategically**—Students have used tables to organize possibilities in the Exploration Staircase Pattern and in Lesson 1, What Can You Say for Sure? This work builds on previous use of tables in Units 1 and 5, where students recorded calculations and built corresponding algebraic expressions.
- **Seeking and Using Structure**—Students have used structure to connect models of staircases in the Exploration Staircase Pattern to corresponding algebraic expressions and to re-contextualize the final algebraic formula. Similarly, students have used the structure of a calculation process to create equations by repeating and generalizing the pattern.
- **Describing Repeated Reasoning**—Students have extended repeated numerical calculations to a variable and recorded each step of the process with algebra in order to create an algebraic representation of the problem.
Before the Unit Assessment, ask students to reflect on the following:

What are some things you learned in this unit?  
What questions do you still have?

Reflections can be done orally, on paper, or some combination of both. Use feedback from students to help them identify the big ideas from the unit and to select Additional Practice problems to help them prepare for the Unit Assessment included on pages T36 & T37. Before giving this assessment, consider spending a class period working through the Unit Additional Practice problems.

Since the Snapshot Check-in, students have:

- Used diagrams to organize information about a problem and determine unknown information.
- Used logic to determine order and the truth value of statements.

Throughout Unit 7, students have focused on the following Algebraic Habits of Mind:

- **Puzzling and Persevering**—Students used logic to reason through distance problems, order puzzles, and Liar/Truth-teller puzzles. They sought entry points to problems and asked what could be determined from information found along the way. The problems in this unit required students to solve problems without having a pre-learned formula or a solving algorithm; they thus support thoughtful perseverance in problem solving.
- **Using Tools Strategically**—Students used number-line-like diagrams to organize information and determine the distances between locations or events in a problem.
- **Communicating with Precision**—Students have interpreted the language of word problems and puzzle clues and worked with “if/then” statements and contradictions to determine a solution. They have also worked to articulate difficult and often multi-step logical reasoning.
- **Describing Repeated Reasoning**—Students have also recorded the steps they took in a multi-step calculation in order to recognize, and then express precisely in mathematical language, a procedure they used by performing that procedure with a variable input.
**Game: Coin Challenge**

**PURPOSE**
This game has students regularly revisit the question “What can be said for sure?” from Lesson 1. You issue a Coin Challenge that gives two clues about actual coins that you have in your hand, and students use logic to determine what you could have by selecting among the various possibilities they are learning to consider. This supports the kind of reasoning students are learning throughout the unit and makes use of new tools and strategies in a tangible scenario.

**Game Suggestions**
Each day, select a different combination of coins, actually have them jingling in your pocket or hand, and offer students both the total value of your coins and the total number of coins. Here are several challenges:

- Prepare 4 nickels and 6 pennies and say:
  “I have exactly 10 coins totaling 26¢ in my hand.”

- Prepare 1 dime, 2 nickels, and 12 pennies and say:
  “I have exactly 15 coins totaling 32¢ in my hand.”

- Prepare 3 nickels and 14 pennies and say:
  “I have exactly 17 coins totaling 29¢ in my hand.”

- Prepare 1 dime, 5 nickels, and 3 pennies and say:
  “I have exactly 9 coins totaling 38¢ in my hand.”

- Prepare 4 nickels and 18 pennies and say:
  “I have exactly 22 coins totaling 38¢ in my hand.”

- Prepare 2 dimes, 3 nickels, and 6 pennies and say:
  “I have exactly 11 coins totaling 41¢ in my hand.”

Give students a few minutes to work the problem out. You can tell students that you will offer this kind of challenge every day, and you may want to institute a system to ensure that everyone gets an equal chance of winning. Students could write their name and solution on a slip of paper, and you draw names until you reach the student with the first correct response.

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**Game at a Glance**

**Timing**
- This game is intended to be done in short 3- to 5-minute sessions throughout Unit 7.
- You may wish to use this game at the end of every class or at the beginning just after the Mental Mathematics. Try extending that introductory time and energy to include a coin challenge.

**Preparation**
- Keep 2 dimes, 5 nickels, and 18 pennies in a safe place for each day’s end-of-class Coin Challenge.

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**Variations**
- You may wish to offer the coins as a prize for the first student to answer correctly.
- You can vary the type of clues given. For example:
  - I have 21¢, and I have 4 times as many of one kind of coin as the other.
  - I have 90¢ in dimes and pennies, and the number of dimes is two less than the number of pennies.
- You may also consider asking students to come up with their own clues or variations on the game.
**Snapshot Check-in**  

**1** Malika has exactly 37¢ in her pocket. She has only nickels and pennies. Write at least three things you can say **for sure** about the coins in her pocket.

**2** What are the possibilities for Malika’s coins?

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You may not need all the spaces in the table.

**3** What is the greatest number of coins Malika can have?

**4** If Malika has 17 coins total, how many pennies does she have?

**5** If Malika has more nickels than pennies, how many coins does she have, in total?

**6** Write and answer two more questions about Malika’s coins.

**7** Write an equation for the number of pennies \((p)\) that Malika has if she has \(n\) nickels. Try some numbers first and use the pattern of the calculations to help write the equation.

\[ p = \]
**Unit Assessment**

1. Jessica has exactly 18¢ in her pocket. She has only nickels and pennies. Write at least three things you can say for sure about the coins in her pocket.

2. What are the possibilities for Jessica’s coins?

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You may not need all the spaces in the table.

3. What is the greatest number of coins Jessica can have?

4. If Jessica has 14 coins total, how many pennies does she have?

5. Write and answer two more questions about Jessica’s coins.

6. Write an equation for the number of pennies (p) that Jessica has if she has n nickels. Try some numbers first and use the pattern of the calculations to help write the equation.

   \[ p = \text{ } \]

7. Asher, Ben, and Carla are each opening a present.
   - They are very polite, and each waits for the person before them to finish before opening theirs.
   - Ben does not open his present first.
   - Carla opens her present after Ben.
   In what order did they open their presents?
Paulo and Damian took a hiking trip the mountains. They climbed from the Base to the Waterfall in 63 minutes. Next they saw the Canyon, and from there to the Lake took only 37 minutes. Their journey from the Base to the Lake took 2 hours and 35 minutes.

How long did it take them to get from the Waterfall to the Canyon?

How long was it from the Base to the Canyon?

You meet two Beebos.
What can you say for sure about these Beebos?

Describe your reasoning on problem 9.

Four buttons are in a row: Yellow, Black, Red, Green.
» Green is to the right of red.
» Yellow is to the left of red.
» Black is not on either end.
» Red is not to the left of black.
What is the order of the buttons?
Distributive property and distance

The aim of the Mental Mathematics activities in the first half of the year was both to generate students’ intuitive sense of important algebraic properties and, with that, to build their arithmetic competence and awareness of that genuine competence. To maintain and strengthen these skills, the activities in the second half of the year must be different and challenging enough to remain interesting, but they must not become so hard that the mental effort is exhausting and makes students feel less competent. Accordingly, the second half of the year includes new content but also considerable consolidation. Along with extensions of earlier work, the Mental Mathematics activities for most of these units also include a “choose a review” option.

The new challenge in the Unit 7 Mental Mathematics activities is for students to combine what they know of multiples of 10 with what they know about complements, distance, fractions, and decimals. For example, students find distance to 30 \( \frac{2}{5} \) or 30.4 using fractions or decimals, respectively.

In nearly all of these Mental Mathematics activities, students “enact a function”: an input-output rule is established at the outset, and students give the output for each input they hear. Each function rule focuses on a key mathematical idea or property (e.g. complements or the distributive property) that students begin to feel intuitively.

After introducing the day’s task, the teacher deliberately does not reiterate the task but says only the input numbers for students to transform. Minimizing words lets students focus on the numerical pattern of the activity, helping them perceive the structure behind the mathematics. A lively pace maximizes practice and keeps students engaged.
Mental Mathematics • Activity 1
Doubling with units digit greater than 4

PURPOSE
Students maintain and strengthen the skill at doubling that they began in Unit 1 and stretched in Unit 5. When we double an input like two hundred five, we can just report back the numbers we “hear” in our heads. With an input like thirty-seven, we don’t report back sixty-fourteen but must instead add 60 and 14 mentally. Today’s activity focuses special attention on that addition step, continuing to build an intuitive sense of the distributive property of multiplication over addition.

Introduce:
“As you’ve done well and often before, you will multiply any number I give by 2. Ready?”

About this sequence:
Input sequences like 20, 8, then 28, help draw attention to the separate doubling of the two parts of 28 and their recombination by addition.

Step 1: Start gently, getting students used to digits greater than or equal to 5.

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Step 2: Shift to prompts in which the units digit remains greater than or equal to 5 and the mental addition step is required. This also includes negatives.

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>85</td>
<td>170</td>
</tr>
<tr>
<td>76</td>
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<tr>
<td>27</td>
<td>54</td>
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<tr>
<td>-46</td>
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<td>68</td>
<td>136</td>
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<td>-78</td>
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<tr>
<td>98</td>
<td>196</td>
</tr>
<tr>
<td>37</td>
<td>74</td>
</tr>
</tbody>
</table>

Some students may comment that for inputs like 49 or 99, an easier mental strategy would be to double 50 or 100 and then adjust. Perfect! Flexibility in thinking about the numbers is exactly what you want.
Mental Mathematics • Activity 2
Doubling, challenges

**PURPOSE**
Today’s purpose is for students to notice how much skill they’ve developed; no new skill, idea, or shortcut is introduced. Let students warm up with about half of the Step 1 inputs below, and then offer them just five of the Step 2 challenges. Leave them hungry for more, not exhausted from the effort or bored with the repetition.

**Introduce:**
“We’re doubling again, but today’s doubling will be very brief—just a few challenges so that you can see how much you can handle in your head!”

**About this sequence:**
Step 2 restricts the tens digits to less than 5 so that the computation does not exhaust students’ mental resources and risk convincing them that they are not good at mental arithmetic. The goal, as always, is to find the “frontier,” a position easy enough to allow success with effort, and hard enough for students to feel that effort was required and that they can therefore feel proud of their success.

**Step 1:** Begin with a short warm-up—at most, about ten inputs (*half this list*).

<table>
<thead>
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<tbody>
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<td>68</td>
<td>136</td>
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<tr>
<td>49</td>
<td>98</td>
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</tbody>
</table>

**Step 2:** Announce that you will give just five challenge problems. Select from these or invent your own with the units digit greater than or equal to 5, tens digits less than 5, and any hundreds digit.

<table>
<thead>
<tr>
<th>206</th>
<th>412</th>
</tr>
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<tbody>
<tr>
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<table>
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</thead>
<tbody>
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<td>1294</td>
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<tr>
<td>-447</td>
<td>-894</td>
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