Dear Student,

A mobile that balances will **stay balanced** as long as the same weight is added to or subtracted from both sides. That same **logic** works for algebraic equations, too.

You've already been applying logic to algebra in Think of a Number tricks and mobile puzzles. In this unit, you will organize and extend that logic to solve algebraic equations, like $2x + 1 = x + 5$, with the same strategies you used to solve mobile puzzles and shape equations like $\star + \star + 1 = \star + 5$.

You'll also learn to think of an algebraic expression as layers of instructions (like a Think of a Number trick). If you can figure out what the steps of the trick are, you can start with someone's final number and then unravel the instructions until you arrive at their starting number.

In this unit, you'll also learn how you can sometimes simplify your work by covering up a complicated expression and (temporarily) pretending it isn't there. Algebra is full of logic; the more you get into the habit of looking for the logic, the more you'll discover.

—The Authors
Lesson 1: Staying Balanced

1. This mobile balances.

2. This mobile also balances.

3. Which of the following changes would keep this mobile balanced? Circle all that apply.

A. Add a pentagon to both sides.
B. Add 5 leaves to both sides.
C. Move all the pentagons to the right side.
D. Switch the leaf and circle.
E. Add a circle to the right side.
F. Remove one pentagon from both sides.

We don’t have enough information to solve any of these mobile puzzles. (Why not?) But we can still talk about the relationships between the shapes.
4 This mobile balances. Use the key to translate it into an equation.

\[ s + \quad = \quad \]

Key:
- $\bigcirc = m$
- $\star = s$
- $\bigotimes = c$

5 Draw the mobile that would remain if you crossed out one star and one moon on each side.

6 Do we know if these changes would keep the mobile balanced? Explain your reasoning.

7 Write an equation for your modified mobile by using the key to translate.

\[ \quad = \quad \]

8 If the equation $s + m + c + m = 2s + m + 2s$ is true, then which of these equations are true? (Mark True or False.)

- (a) $2s + m + 2s = s + m + c + m$
- (b) $4s + m = 2m + s + c$
- (c) $5s + m = 2m + c$

- (d) $s + 2m + c = 4s + m$
- (e) $4s = s + m + c$
- (f) $3s = c + m$

Use this key to choose shapes and translate the mobiles to algebra in problems 9 through 12.

9 This mobile balances. Create a new, balanced mobile based on this mobile.

10 Translate both mobiles into equations.

11 This mobile balances, too. Create a new, balanced mobile based on this mobile.

12 Translate both mobiles into equations.

The same logic about changes that keep mobiles balanced can be used to consider changes that keep equations true.
STUFF TO MAKE YOU THINK

13 Based on this mobile, determine if these equations are true or false.

- \( h + s + s + h = m + m + h \)  
- \( 2s + h = 2m \)  
- \( 6h + 6s = 6m + 3h \)  
- \( 2m + h = h + 2s + h \)  
- \( 3s + 2h = 2m + h + s \)  
- \( 3h = 2m + 2s \)

14 Based on the equation \( t + 2p + 2s = 2t + 3s \), determine if these equations are true or false.

- \( t + 2p + s = 2t + 2s \)  
- \( 3t + 2p = 5s \)  
- \( 2p + 2s = t + 3s \)  
- \( 2p + 5s = 3t \)  
- \( 2p = t + s \)

15 This mobile balances.

Why can’t we find the weight of the shapes on this balanced mobile?

Based on that mobile, can we say for sure that this mobile balances?

Why or why not?

Can we say for sure that this mobile balances?

Why or why not?

Can we say for sure that this mobile balances?

Why or why not?
Fill in the cells in this table.

<table>
<thead>
<tr>
<th>Instructions</th>
<th>Pictures</th>
<th>Abbreviation</th>
<th>Jacob</th>
<th>Mali</th>
<th>Kayla</th>
</tr>
</thead>
<tbody>
<tr>
<td>Think of a number.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Add 5.</td>
<td><img src="mug.png" alt="Picture" /></td>
<td>$b + 5$</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multiply by 2.</td>
<td></td>
<td></td>
<td></td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>Subtract 4.</td>
<td><img src="mug.png" alt="Picture" /></td>
<td></td>
<td></td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

TOUGH STUFF

20. If the equation $2x + 2b = c + b$ is true, then which of these equations are true? (Mark True or False.)
   a. $2x + b = c$
   b. $x + b = \frac{1}{2}(c + b)$
   c. $2x + b - c = 0$
   d. $2x + 3b = c$
   e. $2x + 2b - c = b$
   f. $2x = c - b$

21. Based on the equation $4s + t + m = 2t + 3m$, determine if these equations are true or false.
   a. $4s + 3t = 4m$
   b. $4s + t = 2t + 2m$
   c. $t + m = 2t + 3m - 4s$
   d. $4s + m = t + 3m$
   e. $4s + 2t + 2m = 3t + 4m$
   f. $4s = t + 2m$

22. ![Diagram](image.png)

23. ![Diagram](image.png)
**Additional Practice**

**A** Does this mobile balance *always, sometimes, or never*?

If sometimes, *when*?

**B** Which of the following changes would keep the mobile in problem A balanced?

1. Add 4 green dots to both sides.
2. Remove a bucket from each side.
3. Add a bucket to both sides.
4. Move the dot on the left over to the right side.

**C** This mobile balances.

**I** Can we say for sure that this mobile balances?

Why or why not?

**II** Can we say for sure that this mobile balances?

Why or why not?

**III** Can we say for sure that this mobile balances?

Why or why not?

**D** Use the key to translate the original mobile and each of the balanced mobiles from problem C into algebra.

**E** This mobile balances. Use the key to translate it into an equation.

**F** Imagine crossing out one star and one triangle on each side. Write an equation for that modified mobile.

**G** Based on the mobile above in problem E, determine if these equations are true or false.

1. $2k + 2t = 4s$
2. $2k + s = 3s$
3. $2k + t = 2s + t$
4. $2k = 2s$
5. $2k + t = 4s + t$
6. $k = s$
TEACHING GUIDE
Logic of Algebra

June Mark
E. Paul Goldenberg
Mary Fries
Jane M. Kang
Tracy Cordner

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Heinemann
361 Hanover St.
Portsmouth, NH 03801
www.heinemann.com

Education Development Center, Inc.
43 Foundry Avenue
Waltham, MA 02453-8313
www.edc.org
Unit 5

Logic of Algebra

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MENTAL MATHEMATICS

"Any order, any grouping" (commutative and associative) property of multiplication: combining multiplications and divisions by 2 and 10

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T60Multiplying by 5
T61Multiplying by 5 with more challenge
T62Multiplying by 5 (day 3)
T63Dividing by 5
T64Dividing by 5 (day 2)
Learning Goals

By the end of Unit 5, students should be able to:

- Explore the basic rules of algebraic manipulations by imagining a balanced mobile puzzle.
- View expressions as a series of ordered steps recorded with precise notation.
- Understand, generate, and translate between verbal arithmetic instructions and algebraic expressions.
- Develop mathematical language related to calculations and equations.
- Solve equations using properties of operations and the logic of preserving equality.

Many students experience mathematics as a set of rules to follow. It is tempting to teach rules since they seem more straightforward, and pushing for the underlying logic requires time and effort. Focusing on rules can be faster in the short run, but remembering rules requires either constant maintenance or fundamental understanding, and nearly all of us are better at remembering and applying “commonsense” logic or principles that we understand than rules that we don’t understand.

This unit builds facility with manipulating expressions and equations thoughtfully, logically, and accurately. Students use the commonsense logic they have developed in the Think of a Number tricks and mobile puzzles to make sense of the rules for solving equations. Solving for unknown weights in puzzles about balance builds the same logic as is required in solving equations and systems of equations: seeking every unknown value by performing only operations that preserve balance and substituting equivalent values or shapes as needed.

Mobile Puzzles and Balance

Students have encountered mobile puzzles since Unit 1. Now, they make the explicit connection between mobile puzzles and algebraic equations by relating the solving steps of mobile puzzles to the solving steps in algebraic expressions and by considering which steps maintain balance in these contexts and which may disrupt it. In fact, all of the information from any mobile puzzle can be expressed as a system of equations, and if that system is uniquely solvable, then the mobile puzzle is also uniquely solvable and has the same solution.

Number Tricks and Solving

In Unit 1, Language of Algebra, students learned to create algebraic expressions as a record of the steps of a Think of a Number trick. As they now focus on
the logic of algebra, they see how the instructions in number tricks correspond to the features of a related algebraic expression. Then, given an expression and its value for some unknown “starting number,” students equate the two—thereby creating an equation—and solve that equation by using order of operations to work through the “steps of the trick” backward, undoing the structure systematically to discover the original number, the value of the variable.

Covering Chunks
As students learn to identify the part of an expression that represents the “most recent instruction” of a trick (and thus, the first operation to undo), they learn to treat the rest of that expression as a single “chunk.” Students can then reason about the value of that chunk, create a new equation from that deduction, and work backward from there toward the value of the variable. Students also practice this solving approach with expressions that involve squared terms. An example follows.

Maria thought of a number, added 3, squared the result, divided by 5, and got 20 as her final result. We can use algebra to record this information:

\[
\frac{(n + 3)^2}{5} = 20
\]

We can cover everything but the last step and the final result:

\[
\frac{(n + 3)^2}{5} = 20
\]

Now we ask, “What divided by 5 is 20?”

\[
\frac{100}{5} = 20
\]

We see that what we covered equals 100:

\[
(n + 3)^2 = 100
\]

Then, in that new equation, we cover everything but the last step and the final result:

\[
(n + 3)^2 = 100
\]

Now we ask, “What squared equals 100?”

\[
(10)^2 = 100 \quad \text{and} \quad (-10)^2 = 100
\]

So, what we covered is either 10 or -10:

\[
\begin{align*}
10 & = n + 3 \\
-10 & = n + 3
\end{align*}
\]

We see that Maria either started with 7 or -13:

\[
\begin{align*}
n & = 7 \\
n & = -13
\end{align*}
\]

Mobile Puzzles and Solving
It would be very inefficient (and difficult) to try to solve a system of equations by creating a mobile puzzle, and mobiles don’t really lend themselves to negative or fractional coefficients, so the mobile is only a tool for thinking about solving equations and not at all a method. Nevertheless, the variety of solving steps that can be required by mobiles provides a broadly relevant, fun, informal introduction to (or refresher of) the logic of solving systems of equations.

Consider the following mobile.

Seeking and Using Structure
Students learn to use balance as a metaphor for equality, and they use that metaphor to consider the structural similarity between mobiles and equations or systems of equations. Students also learn how to see the structure of an algebraic expression as the most recent step together with a chunk containing everything else in the expression that represents the result of all operations prior to the most recent step. Students then treat the remaining structure (the “chunk”) as a simpler expression.

Communicating with Precision
As students continue to experience the need to be precise—not only with the vocabulary they use to describe algebraic steps but also with the order in which they present these steps—they begin to develop the habit of communicating with precision.

Using Tools Strategically
Mobiles are useful for visualizing which manipulations maintain balance, but the metaphor is weaker for coefficients that are negatives, fractions, and large numbers. Think of a Number tables demonstrate how to unravel complicated expressions and equations one step at a time but also are not a strategy for solving. These tools are to help students develop, understand, and remember the logic behind the algebra, but ultimately, the algebra itself is the most versatile tool we have.

Puzzling and Persevering
Among the most invaluable features of puzzles are that there are many ways to begin and that they aren’t completed after the first step. Solving puzzles supports the practice of “hanging in there” on multi-step problems.
Now that we know $\triangle = 2$, we can use $\circ = 4 \triangle$ to get $\bigcirc = 8$. And from $\star = 3 \triangle$, we get $\star = 6$. As students puzzle through mobiles and learn the logic of maintaining balance, they build these rules from experience without rushing to formalization. They then transcribe mobiles into algebra to help strengthen the connection. As you help students with algebraic solving, refer to the logic of the mobile: “If you subtract something from one side of a mobile and add it to the other, will the mobile stay balanced?”

**Mystery Number Puzzles**

In these puzzles, students encounter systems of equations in a new form and with a new focus. These systems of “shape equations” are identical even in form to algebraic equations, except that shapes rather than letters are used as variables. Because they focus on algebraic properties (e.g. the “zero property,” the “identity property,” etc.), these puzzles give students systems of equations that are not all linear. Unlike the mobile puzzles, in which students work from known values to discover unknown values, these problems do not offer any known values as clues. Students reason purely from the structure of the equations and the algebraic properties involved. Several clue types appear frequently:

- $\square + \square = \square$
  This clue implies that $\square = 0$.

- $\bullet \star = \star$
  This clue implies that either $\bullet = 0$ or $\star = 1$.

- $\triangle \star = \star$
  This clue implies that $\triangle = 0$ and/or $\star = 1$.

- $\nabla \nabla = \star$
  This pair of clues implies that either $\nabla = 0$ or $\nabla = 2$.

The above chart is included to provide you, as teacher, with a sense of the underlying logic in these tricks. The goal is for students to reason through these tricks and build the logic for themselves. Resist the urge to teach how to solve by memorizing rules. The goal is for students to apply what they know and learn to think logically about equations. If the puzzles in the text seem too hard for some of your students, you may wish to offer just one or two of the relevant equations above without revealing their implications (for example, offering $c \cdot c = c$, but not saying what that clue implies) and ask students to determine the values. Experimentation alone should suffice for nearly all students.

**Mental Mathematics: Multiplying and Dividing by Multiples of 2 and 5**

Students combine and extend previous skills with doubling, halving, and powers of 10 to multiplication with 4 (doubling twice), 20 (doubling and multiplying by 10), 40 (doubling twice and multiplying by 10), and 5 (multiplying by 10 and dividing by 2). These activities support understanding of the “any order, any grouping” property of multiplication (a composite of both the commutative and associative properties), e.g. $x \cdot 10 \div 2 = x \div 2 \cdot 10$, and distribution, e.g. $21 \cdot 4 = (20 + 1) \cdot 4$. 
Students also invert these operations as they divide by 4 (halving twice), 20 (halving and dividing by 10), 40 (halving twice and dividing by 10), and 5 (dividing by 10 and multiplying by 2). These activities support flexibility with mental computation and facility with the properties of operations.

**Explorations**
In the Tiling Patterns Exploration, students develop the habit of organizing the data they collect to guide their search for solutions as they search for all possible ways to fill up a $2 \times 6$ space with $1 \times 2$ tiles.

In the Train of Cubes Exploration, students build an algebraic expression for the number of unit squares on the surface of trains with varying numbers of cubes.

**Related Activity**
As in Unit 1, students create mobile puzzles to solidify their understanding of balance, bolster their intuitive logic about the rules for solving equations, and explore the challenge of building uniquely solvable mobiles.
Lesson at a Glance

Preparation
- Each group of two or three students will need scissors to cut out one set of the Mobile Sorting Cards, available on page 55 of the Student Worktext.
- Prepare to display a set of these cards (also available in the Resources PDF).
- (optional) Provide tape or glue and a sheet of paper divided into YES and NO sections for students to create a poster of their mobile logic.

Mental Mathematics (5 min)
Launch: Mobile Sorting Cards (10 min)
- In groups, students sort the cards into two categories: YES (we know this mobile balances) and NO (we can’t know for sure with the given information).
- Discuss student reasoning for each mobile.

Student Problem Solving and Discussion (30 min)
- After students work through the Important Stuff and explore additional problems, use the discussion questions in this Teaching Guide to solidify student understanding.

Unit 5 Related Activity: Making Mobiles (See page T36 and Student Worktext page 48.)

Lesson 1: Staying Balanced

PURPOSE
Mobile puzzles require attention to balance. If the solving process requires moving or removing objects, those moves must preserve the balance.

Unit 5’s purpose is to help students apply to algebraic expressions and equations the logic they have developed and used fairly naturally in the various puzzle contexts presented so far. Lesson 1 helps students make explicit the logic of solving mobile puzzles and apply that logic to the solving of algebraic equations. The mobile puzzle metaphor serves as a visual mental model for identifying and performing allowable algebraic manipulations. Here, students describe mobiles with algebra and reason about algebra while considering the logic of mobiles. This connects the abstract objects and manipulations of algebra to the tangible context of balancing mobiles.

Launch: Mobile Sorting Cards
Students will sort the Mobile Sorting Cards into piles for YES (we know this mobile balances) and NO (we can’t know for sure with the given information).

As groups work, refrain from correcting individual cards. Instead, give hints like “Three of your group’s cards are in the correct category.” As you circulate, ask questions of both correct and incorrect cards, such as:

- “How did you know for sure that this mobile also balances?”
- “Why isn’t this mobile guaranteed to balance?”

After five to seven minutes, bring the class together even if students are not all finished. Display a set of these cards, and ask one volunteer at a time to answer the question on a card, justify their reasoning, and place the card in the appropriate category.

Use the remaining time to have a student-centered discussion about why some changes keep the mobiles balanced and others don’t. As much as possible, limit your role to recording student responses. If you wish, you may have students tape or glue their cards onto a sheet of paper divided into YES and NO sections.

It is important for all students to experience success but also important to move activities along at an interesting pace. Even fun activities become boring when they drag. Use this card-sorting activity to whet students’ curiosity, and use the student pages to provide further support.
Student Problem Solving and Discussion

Allow students to work through the problems in the Student Worktext together or independently as they like or as you feel appropriate. As they work, help them refine the language they use to describe the changes happening on the mobiles and in the equations. Judiciously ask:

- “How did you know for sure that this mobile also balances?” or “…that this equation is also true?”
- “Why isn’t this mobile guaranteed to balance?” or “Why isn’t this equation guaranteed to be true?”
- “Could this mobile balance? Under what conditions? So, are we sure it’s balanced?” or “Could this equation be true? Under what conditions? So are we sure it’s true?”

The language in several of these problems asks, “Can we say for sure that this mobile balances?” rather than asking, “Does this mobile balance?” Likewise, the answer key uses similarly uncertain language for a problem where we don’t know if the mobile balances: “A star switched sides, so the right side might be heavier.” This language is appropriate because it is possible that the mobile is still balanced if all of the shapes are weightless. Students may not notice this, but if there is time, you may wish to ask, “What has to be true for this mobile to balance?” about either PROBLEM 4 or PROBLEM 7 and see what students come up with.

This mobile balances. Use the key to translate it into an equation.

\[
\begin{array}{c}
\text{G: A kite was added to the left and a circle to the right, and these aren’t necessarily equal weights.} \\
\text{F: A pentagon was removed from both sides.} \\
\text{A: Both sides were doubled (or since } 2\mathbb{H} = m + (c, \text{ both sides increased by the same weight).} \\
\text{B: A pentagon and square were removed from both sides.} \\
\text{D: The sides were swapped and the shapes rearranged, but equal quantities still hang on both sides.} \\
\text{C: A circle and star were exchanged, but they aren’t necessarily equivalent weights.} \\
\text{E: A clover was moved from the left to the right side, so the right side is likely now heavier.} \\
\text{H: A square weight was added to both sides.}
\end{array}
\]

Discussion should follow from the conversation at the beginning of class. Students have had a chance to practice describing the moves as they’ve solved problems, so now is a good time to clarify that, for example, crossing out the same symbols on both sides of the mobiles is equivalent to subtracting their symbols from both sides of a corresponding equation. Emphasize that this is why “algebraic solving steps” work: all steps preserve equality (like balance).
Use questions like these to guide the discussion of moves that preserve balance.

» Why is “doubling” a legitimate operation to make on a mobile? That is, if $3 \star = \bullet + \bigcirc$, can you really say $6 \star = 2 \bullet + 2 \bigcirc$? Aren’t you adding different things to each side? Yes, but those different things represent the same quantity: we know that 3 stars are equal to a circle plus a bucket.

» If $4b = 10c$, do we know $3b = 9c$? No. By what reasoning? On the left, a $b$ was removed. On the right, a $c$ was. This could balance if $b = c$, but then 3 of one can’t equal 9 of the other unless both are 0.

» If $6b = 10c$, do we know $3b = 5c$? Yes. By what reasoning? Students may notice that if the total weight on one side of a mobile is $6b$, then half of the weight on that side of the mobile must be $3b$; likewise half the weight of the other side must be $5c$. Since we know that the two sides are equal, then the two halves, $3b$ and $5c$, must also be equal.

» What changes can you make to a balanced mobile that will keep it balanced? Listen for students who identify the logical steps they use in solving mobile puzzles: adding/subtracting the same thing to both sides, swapping the left and right sides, changing the order of the shapes on one string, doubling/halving/tripling/etc. on both sides, substituting equal quantities, and so on. Students will translate these logical steps on the mobiles into logical steps when solving equations in upcoming lessons.

» What changes to a balanced mobile will make it unbalanced? Here, students can uncover and debug some of the most common errors students make in first-year algebra. Responses may include moving a shape from one side to the other, adding/subtracting unequal amounts, multiplying/dividing shapes on one side of a balanced beam without multiplying/dividing the shapes by the same number on the opposite side of the beam, and so on.
by 2"; the expression \((\frac{2x}{5})\) says “A number is first multiplied by 2 and then divided by 5”; and \(\frac{2}{5}x\) says “A number is multiplied by two fifths.” Given various values for \(x\), which, if any, of these produce the same results? The expressions are equivalent, so they all produce the same results.

» **Build this expression:** Think of a number, add 3, then subtract 1, and then add 8 \((n + 3 - 1 + 8)\). What happens when you switch the order to add 3, add 8, and subtract 1? The corresponding expression \(n + 3 + 8 - 1\) is equivalent to the first. Like a string of operations consisting only of multiplications and divisions, a string of operations consisting only of additions and subtractions can be rearranged and the results will come out the same, as long as values that are being subtracted (or divided) remain connected with their corresponding operation. (A string that mixes additive and multiplicative “steps” cannot be rearranged and produce the same result. Test it out to see.) But notice: “Add 3, and then subtract from 10” is not the same as “Subtract from 10, and then add 3.” Again, the way to know is not by a rule, but by checking it out with numbers or algebra to see for yourself.

» **Show how you can know that the calculations “Add 3, and then subtract from 10” and “Subtract from 10, then add 3” are not the same.** Write both phrases on the board. Considering each calculation numerically with the starting number 7, we can first perform this calculation: \(10 - (7 + 3)\), and then this: \(10 - 7 + 3\). The results of these calculations (0 and 6) are not equal. Using algebra, we start with a variable (perhaps a student’s initial), \(z\), and we get the two expressions \(10 - (z + 3)\) and \(10 - z + 3\). These expressions compute different values and cannot be manipulated into the same form; they are not equivalent.

» **Explain how you can decide whether the calculations “Multiply by 5, and then divide 10 by the result” and “Divide 10 by a number, and then multiply by 5” are or are not the same.** Write both phrases on the board. Reasoning may be similar to this: “Considering each calculation numerically with the starting number 12, we can first perform this calculation: \(\frac{10}{12 \cdot 5} = \frac{1}{6}\), and then this: \(\frac{10}{12} \cdot 5 = \frac{5}{6} \cdot 5 = \frac{25}{6}\). These are clearly not equal. With algebra, we get \(\frac{10}{5z} = \frac{2}{z}\) and \(\frac{10}{z} \cdot 5 = \frac{50}{z}\), which are also not equivalent.” Ask: What’s going on here? Aren’t multiplication and division inverse operations, and aren’t they interchangeable? This focuses on the difference between these two calculations. Because we are dividing 10 by something (and not just dividing by 10), whether the factor of 5 is part of the divisor has an impact on the result. Work out that difference together.

All of these are about translating into algebra the kind of ordinary language that might appear in a word problem or in a “recipe” for a calculation. The “rule” is: When you are unsure, check it out with numbers.
Student Reflections & Snapshot Check-in

Ask students to reflect on their learning by responding to the following prompts:

What are some things you learned in this unit?
What questions do you still have?

Assess student understanding of the ideas presented so far in the unit with the Snapshot Check-in on page T46. You may wish to use the last problem (making up a Think of a Number trick) as optional extra credit. Use student performance on this assessment to guide students to select targeted Additional Practice problems from this or prior lessons as necessary.

So far in Unit 5, students have:

• Connected the idea of balancing a mobile with that of keeping an equation true.
• Translated between mobiles and equations.
• Considered the possible changes that preserve balance on a mobile and keep equations true.
• Recorded instructions algebraically as sets of ordered calculations.
• “Unpacked” the operations in an expression all the way back to the variable signifying the original number.

Students have also focused on the following Algebraic Habits of Mind:

• Seeking and Using Structure—Students have attended to the structure of the kinds of changes that keep mobiles balanced and equations true, and they have used the logic of solving mobile puzzles to solve and manipulate algebraic equations. Students have also explored two methods for recording the steps of a calculation (with simplified and unsimplified expressions) and used the structure of an unsimplified expression to encode and decode the steps of calculations.

• Communicating with Precision—Students have translated between visual mobile puzzles, algebraic notation, and verbal instructions for calculations. Translating between representations supports fluency with mathematical language and notation. Students have also attended to the importance of order in organizing operations; this requires precision in interpreting and generating mathematical language and notation.

• Using Tools Strategically—Students have used mobile puzzles as a tool for investigating the logic of solving algebraic equations. Part of using tools strategically is knowing when not to use them, and here, students have also considered that mobile puzzles are not effective as a method for solving equations, though they are a very good tool for thinking about solving equations.
Snapshot Check-in

This mobile balances. ① Can we say for sure that this mobile balances?

③ This mobile balances. Cross out the two equations that can’t be made from this mobile.

④ Use the record of the final result to complete the table.

⑨ Make up your own trick and write the steps with algebra, as Jay did, so you can see the instructions.

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“Any order, any grouping” (commutative and associative) property of multiplication: combining multiplications and divisions by 2 and 10

These activities combine the commutative and associative properties, which are often confused and rarely necessary to distinguish in beginning algebra. Instead, students learn to rely on the “any order, any grouping” property of multiplication to extend their experiences multiplying and dividing by 10 and 2 to multiplying and dividing by other numbers such as 4, 5, 20, and 40. Doubling twice multiplies by 4. Multiplying by 10 and halving, in either order, multiplies by 5. Multiplying by 10 and doubling, in either order, multiplies by 20. The practice in this unit builds a sense of the way that multiplications can rearrange and combine. Division is handled by treating it as multiplication by the reciprocal, which is commutative. This continues to build students’ ability to hold information in mind, their facility with tracking progress through a calculation, their understanding of multiplicative operations, and their facility with breaking numbers into constituent parts, manipulating, and recombining.

In nearly all of these mental mathematics activities, students “enact a function”: an input-output rule is established at the outset, and students give the output for each input they hear. Each function rule focuses on a key mathematical idea or property (e.g. complements or the distributive property) that students begin to feel intuitively.

After introducing the day’s task, the teacher deliberately does not reiterate the task, but says only the input numbers for students to transform. Minimizing words lets students focus on the numerical pattern of the activity, helping them perceive the structure behind the mathematics. A lively pace maximizes practice and keeps students engaged.
Mental Mathematics • Activity 1

Doubling

PURPOSE
This activity introduces halves to the number that is doubled, requiring students to keep track of the extra “1” generated by doubling $\frac{1}{2}$.

Introduce:
“You’re skilled at doubling integers. For example, if I say 43, you’d say… 86. In this unit, we’ll include halves as well. What if I say 4? Of course, 8, and if I say $4\frac{1}{2}$? Yes, 9. Ready?”

About this sequence:
For 2-digit numbers, units are restricted to values less than 5 (except in the final challenge) in order to simplify calculation and maintain focus on the new challenge of keeping track of the “extra” unit generated by doubling $\frac{1}{2}$.

Step 1: Start with integers, then move to the same integer and a half.

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|-----------------|-----------------|
| 2   | 4               |
| -3  | -6              |
| -8  | -16             |
| 4   | 8               |
| $4\frac{1}{2}$ | 9               |
| -6  | -12             |
| -$6\frac{1}{2}$ | -13             |
| -5  | -10             |
| -$5\frac{1}{2}$ | -11             |
| 7   | 14              |

| Step 2: Use fewer restrictions, only units digits less than 5, and less scaffolding. |
|-----------------|-----------------|
| 34  | 68              |
| 42\frac{1}{2}  | 85              |
| 60  | 120             |
| 64  | 128             |
| $64\frac{1}{2}$ | 129             |
| -74 | -148            |
| -$74\frac{1}{2}$ | -149            |
| 83  | 166             |
| $83\frac{1}{2}$ | 167             |
| 91  | 182             |

Extension: Allow the units digit to exceed 5.

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|-----------------|-----------------|
| 15  | 30              |
| $15\frac{1}{2}$ | 31              |
| 28  | 56              |

| Step 2: Use fewer restrictions, only units digits less than 5, and less scaffolding. |
|-----------------|-----------------|
| 34  | 68              |
| 42\frac{1}{2}  | 85              |
| 60  | 120             |
| 64  | 128             |
| $64\frac{1}{2}$ | 129             |
| -74 | -148            |
| -$74\frac{1}{2}$ | -149            |
| 83  | 166             |
| $83\frac{1}{2}$ | 167             |
| 91  | 182             |

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|-----------------|-----------------|
| 34  | 68              |
| 42\frac{1}{2}  | 85              |
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| $64\frac{1}{2}$ | 129             |
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| -$74\frac{1}{2}$ | -149            |
| 83  | 166             |
| $83\frac{1}{2}$ | 167             |
| 91  | 182             |