Connecting Arithmetic to Algebra

Strategies for Building Algebraic Thinking in the Elementary Grades

Susan Jo Russell, Deborah Schifter, and Virginia Bastable

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Generalizing in Arithmetic

Getting Started, Part I—Noticing

In a second-grade class, students list expressions equal to 15. After students come up with a number of expressions, such as $7 + 8$, $5 + 5 + 5$, and $20 - 5$, the teacher, Louise Craig, adds a constraint:

Ms. Craig: Here’s a trickier one. I’d like you to come up with a number sentence that equals 15 and includes a zero.

Corey: 15 minus zero.

Ms. Craig: How do you know?

Corey: If you minus nothing, you can’t minus anything . . . if you take nothing away the number is the same.

Eduardo: 15 PLUS zero. It’s the same thing, but opposite.

The students in this classroom are engaged in a familiar arithmetic activity—generating expressions equivalent to a particular number. But when the teacher asks Corey how she knows that $15 - 0$ equals 15, Corey makes a general statement about
subtracting zero: when you subtract zero, “the number is the same.” Notice Corey is not specifying 15, but talking about “the number” as if to say she could start with any number, subtract zero, and end up with the same number.

This book examines how instruction in computation can be enriched and deepened by a focus on generalizing about the four basic operations—addition, subtraction, multiplication, and division. It provides examples of how teachers in grades 1–6 can incorporate such a focus into their mathematics instruction to strengthen all students’ fluency with and understanding of the basic operations. Included are accounts of how both students who excel and students who struggle with grade-level computation can benefit from this work.

GENERALIZING ABOUT THE OPERATIONS—A FOUNDATION OF ARITHMETIC

Children spend much time in mathematics solving individual problems. But the core of the discipline of mathematics is looking across multiple examples to find patterns, notice underlying structure, form conjectures about mathematical relationships, and, eventually, articulate and prove general statements. Once Corey has brought up the idea that subtracting zero from any number results in that same number, the class has an opportunity to work on this idea explicitly. Corey’s idea is not about the number zero, but about how zero behaves as part of a particular arithmetic operation. Corey and Eduardo are identifying an important property of addition and subtraction. Later Deirdre comments, “I think times zero is the same thing.” As the class continues this discussion, they will find that adding zero to or subtracting zero from a number results in that number, but multiplying a number by zero does not. Deirdre’s conjecture is incorrect, but her idea allows the class to notice how addition and subtraction differ from multiplication.

These second graders are on a journey to understanding the foundations of arithmetic—how the operations behave, what their properties are, and how they are related to each other. These students are starting to think about the general ideas underlying arithmetic.

We study mathematics, in part, so that we can solve problems in daily life. But mathematics is also a way of thinking that involves studying patterns, making conjectures, looking for underlying structure and regularity, identifying and describing relationships, and developing mathematical arguments to show when and why these relationships hold. We use ideas about such mathematical relationships to solve problems, often without noticing. For example, suppose you are standing in a store, figuring out your change from a $10 bill for two items you are buying. One costs $2.49 and one costs $4.99. (Pause here for a moment before reading on and think through how you would solve this problem mentally.)

One method you might use is: $2.49 is almost $2.50, and $4.99 is almost $5.00. $5.00 plus $2.50 is $7.50. If the cost were actually $7.50, the change from your $10.00 would be $2.50, but you added $.02 to the sum of the actual prices. Your actual cost is
2¢ less, or $7.48, and therefore your change should be 2¢ more, or $2.52. This method uses a general idea about the operation of addition, the claim that these two expressions are equivalent:

\[ 7.50 + 2.50 = 7.48 + 2.52 \]

The person making this calculation probably doesn’t explicitly think about why these two expressions are equivalent—the reasoning involved may be virtually automatic. We could write down the underlying generalization explicitly as follows:

If you subtract an amount from one addend and add the same amount to another addend, the sum remains the same.

\[ 7.50 + 2.50 = (7.50 - 0.02) + (2.50 + 0.02) = 7.48 + 2.52 \]

Students learn procedures that are based on just such generalizations, but they may learn them only as steps that work, without understanding why the steps make sense. In the elementary grades, some students notice and use these kinds of ideas in their computation all the time, but too often, generalizations are not made explicit for the class as a whole. By focusing on making and justifying generalizations in the context of arithmetic, students are supported in building a more complex grasp of the operations. This knowledge will build computational fluency and, in later grades, facilitate the transition from arithmetic to algebra.

**THE FIRST STEP FOR THE TEACHER: NOTICING**

Generalizing about an arithmetic operation may be a new and unfamiliar focus for you and your students. But generalizing in the context of arithmetic is not extra content. Rather, students are already using general ideas about the operations as they solve addition, subtraction, multiplication, and division problems. These generalizations may come up implicitly or explicitly as students observe patterns or discover regularity. You may notice students using a rule without explicitly stating it, or you may hear students express particular generalizations and use them. For example, a student working on addition problems might say, “I know that 6 plus 4 equals 10, so 4 plus 6 equals 10—you can turn the numbers around and it makes the same thing.”

Paying attention to opportunities for generalizing that emerge from students’ work is the first step in making these important ideas a regular part of instruction. What generalizations are expressed during students’ computation? What underlying generalizations are students using without thinking about them explicitly? Looking at your students’ work through this lens can help you identify generalizations about operations that might be fruitful for your class to investigate. As you read the following example, consider how one grade 5 teacher sets up an activity in which students are likely to notice patterns, notes what students notice, then builds on one idea that comes up.
Grade 5

Near the beginning of the year, Marlena Diaz engages her fifth graders in a Number of the Day activity. She asks students to write a list of addition expressions equivalent to 32 in their math journals. She deliberately picks a number that her students can work with easily because her focus is not on the computation itself but on generalizing about addition. As she walks around the class observing students’ writing, she notices that Sean has the following written in his journal:

\[
30 + 2 \\
29 + 3 \\
28 + 4
\]

She posts his sequence on the board, and asks students to consider it.

Christy: We can keep going. Add “27 plus 5.”

Lateia [coming up and pointing]: See these keep going down and these keep going up.

Ms. Diaz: Why do you think that is happening? Talk in your table groups for a while.

After a few minutes of student discussion:

Ms. Diaz: So what do you think is happening?

Kathryn: Well, you take some from one number and give it to another.

Amelia: All that is happening is that you are moving some amounts around. And it stays the same.

Christy: You aren’t adding any or taking any away.

Eddie: Right. It is like if you had two groups of dots. We just are changing the size of the groups.

Sean: And since all numbers are made up of ones, we can just move all those ones around.

Jonah: And we are going to get to a point where it won’t work anymore. When we get to 16 plus 16.

Sean: Right, then it will start all over again. It will repeat but basically it is the same numbers switched around.

In the grade 5 class, as students look at Sean’s list, they begin to describe what is consistent in all the expressions:
“You take from one number and give it to another.”

“You are moving some amounts around. And it stays the same.”

“You aren’t adding any or taking any away.”

The students at first notice the pattern in their series of expressions: 30 + 2, 29 + 3, 28 + 4, and so on. Lateia observes, “These keep going down and these keep going up,” referring to the two columns of numbers. The conversation could have remained at the level of making observations about what the pattern looks like. But the teacher asks her students, “Why do you think that is happening?” to focus them on reasoning about the relationship among these expressions.

The discussion is just beginning for these students. They have much engaging work to do as they investigate further. Eddie’s idea about visualizing two groups of dots will provide a mechanism for the class to explain why this generalization works. Representations such as drawings, models, number lines, groups of objects, rectangular arrays, and so forth are the tools that are available to young students for reasoning about general claims they are making about an operation. As students continue their discussions, the teacher will work with them to articulate a clear statement of their idea, such as the one articulated earlier for the problem $4.99 + 2.49$. (Look back at that problem and think about its relationship to these fifth graders’ ideas.) They will represent their idea in a variety of ways and move toward proving that their idea holds true for any addition problem. They will develop a strong foundation for applying general rules with understanding in their computation work and, later, in the realm of algebra.

The first step, then, is to notice general ideas that occur in the course of your regular computation instruction. Here are two brief examples of students solving arithmetic problems at the beginning of the year. As these teachers get to know their students and how they think about numbers and operations, the teachers also notice generalizations that are either implicit or explicit in students’ work. As you read the examples, keep these questions in mind:

1. What relationships between two arithmetic expressions are students noticing?
2. If you were the teacher in one of these classrooms, what questions might you ask to help students pursue these beginning ideas further?

Grade 1

Emma Perkins is working on an activity called “How Many of Each?” with her class.1 The problem she poses is, “You have 10 vegetables on your plate. Some are

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peas and some are carrots. How many peas and how many carrots could you have?” Ms. Perkins writes:

Each child counted out beans, which were green on one side and orange on the other. All of the students settled on $5 + 5 = 10$ as the answer. I got blank faces when I asked if there was another way to solve the problem. It seemed as if these children were used to getting one answer and moving on to the next part of the lesson. But I wasn’t ready, and I could wait a long time. After repeating my question for a different solution, Harriet played with her beans and said “9 plus 1 equals 10,” which she wrote on her paper. Mikel then decided that $7 + 3 = 10$ would be another answer. Connor complained, “I was going to write that.” I suggested that they both could put it down and waited to see what might happen. As it turned out, Mikel wrote $7 + 3 = 10$ and Connor wrote $3 + 7 = 10$, creating a great opportunity for comparing these two expressions.

Grade 4

At the beginning of the year, Marie Taft establishes a routine called “What Do You Know About ___?” She puts an expression on the board and asks students to tell her what they know about the number it equals without doing any computation. Like Marlena Diaz, Ms. Taft uses numbers that are easily accessible to her students so that they can focus on relationships between expressions. The previous day, students had discussed the expression $4 \times 10$. In order to build on that discussion, Ms. Taft writes $4 \times 20$ on the board. Here is part of the conversation she documented:

Ms. Taft: What do you know about $4 \times 20$?

Jasmine: I know it is even just like $4 \times 10$ because both of the numbers are even, just like $4 \times 10$.

Faith: I know that it is going to be more than $4 \times 10$ because $20$ is double $10$.

Ms. Taft: Can you tell me more about that?

Billy: It is just double $4 \times 10$ because $20$ is double $10$. It is like if you give $4$ kids $10$ pencils, and then give them $10$ more, you have given them $20$, and $20$ is double $10$.

Most hands shot up at this point, so I had partners talk to each other about it again. When the discussion lightened, I asked someone to explain what Faith meant.

Billy: It is just double $4 \times 10$ because $20$ is double $10$. It is like if you give $4$ kids $10$ pencils, and then give them $10$ more, you have given them $20$, and $20$ is double $10$.

In these two examples, teachers are learning what their students know at the beginning of the year. As part of this work, the teachers set up situations in which
they can pay attention to what students notice about the behavior of the operations. What might these teachers be noticing?

In the grade 1 “How Many of Each?” activity, students generate addition expressions equivalent to 10. Given more such opportunities, will students notice that 3 + 7 and 7 + 3 are both equivalent to 10? Will they notice other pairs of expressions, such as 1 + 9 and 9 + 1? Will they conjecture that changing the order of the addends does not change the sum, not just for 10 but for any number (the commutative property of addition)?

In the grade 4 episode, students notice something about the relationship between 4 × 10 and 4 × 20. One student knows that the product of 4 × 20 will be greater than the product of 4 × 10 “because 20 is double 10.” Another conjectures about how much more 4 × 20 is than 4 × 10. Billy makes an important contribution, offering an image to support his conjecture, “It is like if you give 4 kids 10 pencils, and then give them 10 more, you have given them 20, and 20 is double 10.”

Will students extend their thinking beyond the particular numbers to notice that doubling one factor in any multiplication expression doubles the product? With additional focus on this idea, will students be able to articulate and demonstrate why this occurs? Will they become interested in investigating what happens to the product if both factors are doubled?

Ms. Perkins and Ms. Taft are noticing ideas their students might pursue in the context of the year’s work on computation and operations. Ms. Perkins notices an idea implicit in students’ work that the students themselves may not yet have recognized. Ms. Taft notices some students articulating an idea that can lead to a conjecture about multiplication. Ms. Diaz leads a discussion in which students start to articulate a general idea about the operation of addition. The general ideas you notice in your students’ work become the starting points for engagement in generalizing about the operations.
FOCUS QUESTIONS

GENERALIZING IN ARITHMETIC: GETTING STARTED, PART I—NOTICING

1. The first section of Chapter 1 makes the case that elementary math study should include opportunities for students to investigate general ideas about the operations they use in their computation work. As you reflect on this passage (pages 1–3), consider the following questions.
   • What examples from your own classroom come to mind?
   • Do these paragraphs suggest shifts you might want to try in your own practice?
   • What questions does this passage raise?

2. The beginning of the second section of Chapter 1 states that having students generalize in the context of arithmetic should not be seen as extra or additional content but as a regular focus of the class, enhancing the work already included in the curriculum.
   • What are your thoughts about that statement?
   • What are the implications of that statement?
   • How might that work in your classroom?

3. Consider the student discussion from Marlena Diaz’s class as an opportunity to analyze student thinking and teacher moves.
   • What does Lateia offer the class?
   • What are the contributions of Kathyrn, Amelia, Christy, and Eddie?
   • Explain what Jonah and Sean notice.
   • Examine the actions and questions of Ms. Diaz. Consider what her purpose might be for each of these.

4. Consider the examples from Emma Perkins’ grade 1 and Marie Taft’s grade 4.
   • In each example, what are the arithmetic expressions to be compared? What generalizations might be examined on the basis of these comparisons?
   • If you were the teacher in one of these classrooms, what questions might you ask to help students pursue these beginning ideas further?
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