Contents

Foreword ........................................................................................................ v

Acknowledgments ......................................................................................... ix

Introduction ...................................................................................................... xi

CHAPTER 1  Number and Operations ................................................................. 1
  Counting with Number Words ..................................................................... 2
  Thinking Addition Means “Join Together” and Subtraction Means “Take Away” ................................................. 7
  Renaming and Regrouping When Adding and Subtracting Two-Digit Numbers ................................................. 13
  Misapplying Addition and Subtraction Strategies to Multiplication and Division ............................................. 20
  Multiplying Two-Digit Factors by Two-Digit Factors .................................................................................... 24
  Understanding the Division Algorithm .......................................................... 29
  Understanding Fractions .............................................................................. 34
  Adding and Subtracting Fractions ................................................................. 40
  Representing, Ordering, and Adding/Subtracting Decimals ............................ 43

CHAPTER 2  Algebra ........................................................................................... 49
  Understanding Patterns ................................................................................. 50
  Meaning of Equals ........................................................................................ 55
  Identifying Functional Relationships ............................................................ 61
  Interpreting Variables ................................................................................. 66
  Algebraic Representations ........................................................................... 71
### CONTENTS

#### CHAPTER 3  **Geometry** .......................................................... 78
- Categorizing Two-Dimensional Shapes .......................... 78
- Naming Three-Dimensional Figures ............................. 84
- Navigating Coordinate Geometry ................................. 88
- Applying Reflection ....................................................... 95
- Solving Spatial Problems .............................................. 100

#### CHAPTER 4  **Measurement** .................................................. 108
- Reading an Analog Clock ............................................ 108
- Determining the Value of Coins .................................. 116
- Units Versus Numbers ................................................ 121
- Distinguishing Between Area and Perimeter .................. 127
- Overgeneralizing Base-Ten Renaming .......................... 132

#### CHAPTER 5  **Data Analysis and Probability** ......................... 137
- Sorting and Classifying ............................................... 137
- Choosing an Appropriate Display ................................ 142
- Understanding Terms for Measures of Central Tendency .. 148
- Analyzing Data ......................................................... 152
- Probability ............................................................ 158

#### CHAPTER 6  **Assessing Children’s Mathematical Progress** ......... 164
- Assessment: The Received View .................................. 164
- Assessment from a Better Angle .................................. 165
- Types of Formative Assessments ................................. 166
- Why Assess? ........................................................... 170
- Final Thoughts ....................................................... 171

#### References ................................................................. 173

#### Index ................................................................. 179
Foreword
STEVEN LEINWAND

Nearly all of our students make mathematical mistakes—often perfectly logical mistakes based on common misunderstandings. Nearly all of our students are sometimes confused—often in very understandable ways that emerge from their efforts to make sense of new material. Effective teachers have always understood that mistakes and confusion are powerful learning opportunities. Moreover, they understand that one of their critical roles is to anticipate these misconceptions in their lesson planning and to have at their disposal an array of strategies to address common misunderstandings before they expand, solidify, and undermine confidence. And when, despite our best intentions, misconceptions present themselves, effective teachers are ready with an array of approaches that address these misconceptions head-on before they fester into serious disability.

Some of these common misconceptions are rather familiar:

■ \( \frac{3}{8} + \frac{2}{8} = \frac{5}{16} \) because “you add the tops and add the bottoms”
■ .23 > .4 because 23 > 4
■ if 14 = ____ + 7, then 21 goes in the blank because 14 + 7 = 21
■ circles can’t have areas of square inches or square centimeters
■ squares are not rectangles because “they have four equal sides”
■ \( \frac{1}{2} > \frac{1}{3} \) because 2 < 3
■ the probability of spinning a 1 on the spinner shown is \( \frac{1}{6} \) because there are three numbers or \( \frac{1}{5} \) because there are five segments
Others types of misconceptions are less familiar:

- decimals get smaller as they grow longer to the right of the decimal point
- graph scale errors
- telling time
- measurement conversion errors
- graphing points in the coordinate plane
- confusing expected vs. experimental results
- three-dimensional spatial visualization

Too often, however, we tend to focus overwhelmingly in mathematics classrooms on right answers and correct processes to get these right answers. Too often, wrong answers are met with a simple, but not very helpful, “no” or “wrong” or a big red X, instead of being probed for the source of the error or underpinning of the misconception. And while some errors are certainly careless or based on a lack of fundamental knowledge (for example, confusing area and perimeter), many student errors arise from one or more misconceptions—misconceptions that arise so often they can almost be catalogued. Consider the perfectly understandable direction to “add a zero” when multiplying by ten being interpreted by students as “adding nothing” to the number! No wonder they are so confused when we magically change 34 to 340, explaining that “we added a zero.” In other words, so often, students aren’t “wrong” so much as they don’t “get it” or they’re overgeneralizing, or they’re misled by words or rules that don’t apply.

In this wonderfully insightful book, Honi, Christine, and Karren not only describe a vast array of perfectly understandable mathematical misconceptions that students have across the elementary curriculum, they also provide a brief research basis that gives a context for these misconceptions. But most helpfully, they provide an array of practical instructional strategies and activities for helping remediate and undo the misconceptions. The classroom vignettes they describe...
will ring true to everyone who has tried to teach mathematics to young, and not-so-
young, children. Experienced teachers will smile and recall that it was Kyle or Ali-
cia who “used to do that.” New teachers will recognize that the incorrect
explanation that Sarah provided yesterday is really pretty typical among students
learning mathematics.

Through the realistic, and frighteningly familiar, classroom vignettes that are
provided for each mathematical topic, teachers will discover the power of classroom
norms in which children are expected to explain their thinking as a fundamental
classroom practice for exploring students’ understanding and identifying possible
misunderstanding. The vignettes realistically capture both teacher questioning and
student thinking—on-target and not as on-target—in ways that transport the reader
into the world of real students in real classrooms wrestling with important mathe-
matical ideas.

But the real power of this book lies in the example-laden Ideas for Instruction
that follow the vignettes and accompany each of the topics that are explored. Here
one finds ideas for activities that provide opportunities for students to represent,
construct, visualize, draw, connect, explain, and describe. In other words, the key
message is that we can prevent or minimize many common misconceptions and
effectively address those that still emerge when our instruction consistently probes
students’ understandings and provides opportunities for students to show and
explain their reasoning. That’s the type of mathematics instruction every student
deserves and that this book advocates and describes.

So while misconceptions serve as the organizing spine of this book, the subtext
is providing effective, high-quality mathematics instruction at all times. Of course
it is essential that teachers recognize and anticipate misconceptions and even under-
stand the research findings that help to explain these misunderstandings, but it is
the instructional tasks, the ongoing classroom discourse, and the embedded form-
ative assessment—all components of good instruction and the activities that com-
prise the “Ideas for Instruction”—that make the real difference in student learning
of mathematics.
Introduction

Making sense is at the heart of mathematics and so it must also be at the heart of the mathematics we do with young children.
—Kathy Richardson

Children enter prekindergarten filled with ideas about numbers, shapes, measuring tools, time, and money. They formulate these ideas through their visual and auditory experiences—the expressions they hear adults say and the things they see on television, computer screens, in children’s literature, and all around them. It’s no wonder that some children develop very interesting and perhaps incorrect ideas about mathematical concepts.

“I can count to a million,” a child says as he works with classmates at a number center. “I can count to a million zillion,” responds his friend, naming an even bigger number.

Once prekindergartners have learned to identify all ten digits in the numeration system, they often try to make sense of time on a digital clock. “It’s two dot dot one four,” a four-year-old might say. While they may have no idea what this means, they are making sense of what they’ve heard about numerals and shapes. And, third through fifth graders seem to repeat what they hear all the time from adults who mispronounce numerals. It’s not uncommon to hear a student read the numeral “2010” as two thousand and ten, even though it should be read as two thousand ten. Often what they hear is what they say.

How can we, as early childhood and elementary educators, connect the informal knowledge that students bring to our classrooms with the mathematics program...
adopted by our school system? Just as important, how do we ensure that the mathematics we are introducing and reinforcing is accurate and appropriate and will not need to be retaught or reexplained in later school years?

What We See in Classrooms

In our travels to classrooms across the country, we see teachers who care about their students. Men and women who decide to work with prekindergarten through fifth-grade students do so because they enjoy working with young children and want to help them develop a strong foundation for learning. By and large, these educators are nurturing and supportive and have students’ best interests in mind. They want their students to enjoy learning and end the academic year with all the skills and concepts necessary for continued success.

But these positive characteristics can sometimes lead teachers to unwittingly encourage serious error patterns, misconceptions, and overgeneralizations on the part of young learners. For example, one of the big ideas developed in a typical first-grade classroom is making sense of addition and subtraction concepts and learning strategies to obtain basic fact fluency. It is not unusual to hear a classroom teacher say, “Now, remember boys and girls, we always subtract the smaller number from the larger number.” Students see a story problem such as Maria has 6 candies. She gives her friend, Juan, 2 of her candies. How many candies does Maria have now?, look at the “2” and the “6,” think which is the lesser, and subtract the lesser number from the greater number, giving them the correct difference.

The problem is, young children do not always differentiate between numeral, number, and digit. At the end of first grade or at the beginning of second grade, when they begin to use two-digit numbers, students see 23 – 4, remember what the teacher has said, take the smaller digit away from the greater digit, and end up with a difference of 21.

Does this error occur only because of what some teachers have said? Of course not. There are many other reasons students subtract in this manner. But in “helping” students make sense of subtraction with seemingly innocent supports, some teachers inadvertently create more problems. And it isn’t true that you “always take the smaller number away from the larger number.” In sixth grade, when positive and negative integers are introduced, students will learn of many situations when a greater number is subtracted from a lesser one.

Do these misconceptions and error patterns occur only in the earliest grades and only with number and operations? Absolutely not! Many students think that all hexagons are yellow and have six sides and angles that are exactly the same size,
because the only time they see hexagons is when they are using pattern blocks. This overgeneralization naturally causes problems when these students are asked to create a hexagon, each side of which has a different length.

Commercially made posters available at many teacher supply stores can also support students' misconceptions and overgeneralizations. Many children think that a rectangle has to have "two long sides and two short sides," because these are the only examples they see. This becomes a problem in later grades when they are asked to classify a variety of shapes or are told that all squares are rectangles. Many children just don't believe this last statement (and there are adults who don't believe it, either!). Many intermediate students do not believe there are any numbers between zero and one. Few believe there are any numbers between .1 and .2. The examples they have been given (or not given) contribute to these misconceptions.

What You'll Find in This Book

We identify many common errors relative to the five National Council of Teachers of Mathematics (NCTM) content standards (NCTM 2000) and investigate the source of such misunderstandings. If the problem is the result of something said or shown, we propose how to respond and suggest alternate ways to teach the concept so that the misconception can be avoided. If the misconception or error pattern is already ingrained, we share ideas and activities that help "undo" the confusion.

Chapters 1 through 5 delve into the types of misconceptions that children have across each of the five NCTM content standards: number and operations, algebra, geometry, measurement, and data analysis and probability. In these chapters, you will see how things we say and do (or don't do) may impact student understanding. Each chapter includes several classroom vignettes, each of which highlights a common misconception for that particular content area. Following each vignette, we point out the error being made and present some research providing you with a deeper understanding of how commonplace this error is. We then offer reasons for the error and provide you with numerous instructional ideas and activity suggestions that may prevent or remedy the misconception. Finally we pose questions for you to think about on your own or to discuss with colleagues. It is our hope that these questions will guide your reading of this book, allowing you to discuss not only these errors but others your own students are making and why this might be.

Chapter 6 offers assessment ideas (formative and summative) that can be used to quickly assess a student's level of understanding and potential for developing an error pattern or misconception.
Additional Resources

In addition to this book, we have written two activity books—Activities to Undo Math Misconceptions, Grades PreK–2 for teachers of preK through grade 2, and Activities to Undo Math Misconceptions, Grades 3–5 for teachers of grades 3 through 5. Each of these books contains teaching suggestions as well as black line masters for more than seventy-five activities (some are referenced in the book you are currently reading, some are not). These activity books give you a concise glimpse at an error pattern and offer you a list of suggestions for eliminating it. Each activity book also comes with a CD-ROM containing editable versions of all of the activities in English and in Spanish. This feature allows you to take our ideas and make them more appropriate for your own students, either by differentiating the level of difficulty or making the situations more relevant to your particular students.
A
n entire book could be written solely on the overgeneralizations, misconceptions, and error patterns that early childhood and elementary students have about number and computation. Error Patterns in Computation, by Robert Ashlock (1994), is just such a book. It presents several examples of a series of error patterns, then asks the reader to figure out what the student is doing (come up with a diagnosis) and think about what a teacher might do to remediate the student's thinking.

Our chapter begins with counting, because children in prekindergarten and kindergarten often forget to include the “teen” designations (thirteen, fourteen, fifteen, and so on). Also, many teachers of young children relate addition to “joining” and subtraction to “separating” and forget that there are other models for each operation. As a result many children get confused when they are asked to find the difference between two sets of objects, answering the “how many more” question with the number of objects in the set that has more. Similarly, open sentences or part-total ideas create problems because they haven’t been addressed in earlier instruction.

There are very few second-, third-, and even fourth-grade teachers who don’t complain about students’ inability to add and subtract multidigit numbers (especially when it involves regrouping and renaming). It’s the biggest concern we hear from second-grade teachers who are responsible for introducing these operations to their students. And it’s what so many third-grade teachers say they would love students to begin third grade knowing how to do.

Multiplication and division are next. While these operations are predominantly taught in third through fifth grades, younger students are introduced to the ideas
behind these areas of computation in story problems. Yet students seem to have a great deal of confusion about how to model both operations and then how to make these computations with multidigit numbers.

And then there are fractions. We could easily have written a dozen segments on the problems that elementary students have understanding what a fractional amount means, forming equivalent fractions and simplifying them, and adding and subtracting fractions. The first fraction that prekindergarten and kindergarten students are exposed to is one-half—a concept that isn’t hard for a young child to grasp. It’s the introduction of the symbols, too early, that seems to lead to some strange “understandings.” Here we deal with forming equivalencies and simplifying to lowest terms, as well as with adding and subtracting fractions.

Finally, we couldn’t present a chapter on number and operations without looking at decimal numbers. First graders learn how to write monetary amounts using a decimal point, and by fifth grade, students are adding and subtracting decimals with different place values.

Counting with Number Words

It’s the second week of school in a public school in Newark, New Jersey. The kindergarten teachers are doing all sorts of counting activities to determine which students can count in a stable order, which can count rationally (and up to what quantity), and which have “emerging” skills.

In one classroom, Seth is “showing off” his counting skills by determining the number of plastic bears in a bowl. “I can count all these bears!” he announces proudly.

“Let’s see how many there are,” his teacher responds.

“One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, sixteen, seventeen, eighteen.”

“So, how many are there?”

He smiles and says, “There’s eighteen.”

“Show me again how you figured that out.”

As before, Seth counts each bear, picking them up one at a time, and skips over the number fifteen.
Thanking Seth, the teacher asks Marta to figure out how many bears she has in front of her.

“One, two, three, four, five, six, seven, eight, nine, ten, one-teen, two-teen, three-teen, four-teen, five-teen, six-teen, seven-teen, eight-teen, nine-teen.” She stops there, even though there are more bears.

“What comes after that?”

“I don’t know those numbers yet,” Marta says.

Liam has ten bears neatly lined up in front of him. His teacher watches as he points to each one: “One, two, three, four, five, six, seven, eight, nine.”

“How many bears do you have there?”

“Nine,” he says, smiling.

Perhaps the funniest sequence is that of a student who seems extremely confident. She’s taken several handfuls of connecting cubes out of the large bag on her table, challenging herself to count to a high number. “One, two, three, four, five, six, seven, eight, nine, ten, tenty-one, tenty-two, tenty-three, tenty-four, tenty-five, tenty-six, tenty-seven, tenty-eight, tenty-nine.” Here she stops, even though there are a few more cubes in front of her.

Identifying the Error Patterns

There is no pattern to be found in the number words one through twelve. Then when students begin counting the “teens,” thirteen and fifteen are confusing, because they base their prefix on the ordinal numbers third and fifth. All the other teen numbers have the cardinal number word in front of the teen (for example, fourteen). Seth skipped fifteen in his number word sequence, counting the set of seventeen plastic teddy bears using one-to-one correspondence, verbally tagging each object with a number word but skipping fifteen. When his teacher asked “So, how many are there?” he responded without recounting, “There’s eighteen.” This demonstrated that he was developing an understanding of cardinality—that the last number said also represented all those counted before—even though his count was not accurate. When asked to recount, he skipped fifteen again.

Marta was satisfied with the counting sequence she used because it sounded right. She had connected the words from eleven to nineteen in a pattern that made
sense. Since all these number words end in *teen*, she overgeneralized the counting pattern and put each number word (from one to nine) in front of the teen numbers, using *thirteen* and *fifteen* because they sounded like the others.

Liam counted ten teddy bears and got nine. He thought that *se* was one number word and *ven* was the next. His misconception was that number words are one syllable. The number words one through six all have one syllable. And so do eight, nine, and ten. Therefore, his count stopped at nine.

We've all heard children who skip over number words and even return to number words already recited in order to continue their counting sequence. They remember some number words at the beginning of a count, forget some later, and skip ahead to others as their count continues. These students know there is a “word” for each object counted, because they hear and watch their classmates model this very thing daily. However, their inconsistency hinders their ability to count rationally.

What the Research Says

Research into young children’s understanding of number by the American psychologists Rochel Gelman and C. R. Gallistel (1986) revealed five principles of counting:

1. **One-one principle.** Each item to be counted has a “name,” and we count each item only once during the counting process.

2. **Stable-order principle.** Every time the number words are used to count a set of items, the order of the number words does not change.

3. **Cardinal principle.** The last number counted represents the number of items in the set of objects.

4. **Abstraction principle.** “Anything” can be counted and not all the “anythings” need to be of the same type.

5. **Order-irrelevance principle.** We can start to count with any object in a set of objects; we don’t have to count from left to right.

Ideas for Instruction

A great numbers of books can be used as a springboard for counting activities that enrich students’ understanding. The preK–grade 2 and grades 3–5 activity
books each include a bibliography of our favorite counting books. Below are some recommended instructional activities and strategies for supporting children’s developing ideas about counting, whether they are done with preK students or with students in fourth and fifth grades as they count by fractions and decimal numbers.

▶ Count out loud often with students. Children count higher when they count together, and they will hear the number word sequence being used correctly.

▶ Lead children in singing counting songs and chanting counting rhymes. These engaging experiences help them develop rote counting skills and understand the stable-order principle.

▶ Match the strategy of counting out loud with concrete objects. For example, count the number of students in class by tapping a student on the shoulder as each number word is said. This strategy supports students’ understanding of the one-one and stable-order principles.

▶ Use a circle counting game to help students with the more difficult numbers of eleven, twelve, and those in the teens. Arrange students in a circle and designate a number such as twelve. Students begin to count by ones, and the student who says twelve sits down. The next student begins to count from one again; again, the student who says twelve sits down; and so on (Wright et al. 2006).

▶ Expect your students to count from a number other than one. For example, ask students to count by ones beginning at six. Initially they may need to whisper one, two, three, four, five before going on. Many opportunities to practice will strengthen their ability to count on immediately rather than beginning at one each time.

▶ Promote counting on by having students toss a pair of dice and “total” the two numbers. For example, if the number of dots, or pips, showing on the top face of one die is five and the other is three, students count on from five saying, “six, seven, eight” (three more numbers) or from three saying, “four, five, six, seven, eight” (five more numbers) to get the total number of pips.

▶ Provide opportunities for students to count backward using the number word sequence in reverse. Instead of always counting backward from ten, let students count backward from other numbers such as eight.
Help students understand the one-one principle of counting by:
- Letting them use salad tongs to pick up each object as it is counted.
- Letting them use a stick to point to each object being counted.
- Giving them an egg carton or ice cube tray and having them place each object being counted in the individual spaces.

Help students see the need for keeping track of the individual items being counted. (They should count concrete objects, which they can move, rather than pictures on worksheets, which cannot be moved.) Young children often use the correct number word sequence and tag each object with one-to-one correspondence, but count some of the objects twice, forget to count others, or keep on counting because they haven’t kept track. Give students “counting mats” that have a line drawn down the middle so that as each object is counted, they can slide it across the line to the other side. Placing the counted objects in a paper bag reinforces not only keeping track but the stable-order and one-one principles as well. Egg cartons and ice cube trays also help students keep track of objects as they are counted.

Help students understand the cardinal principle by asking, “How many?” when students complete a counting task. Another strategy is to count a set of objects together, and at the end of the count repeat the final number: “One, two, three, four, five. There are five books.”

Use number-logic riddles to prompt students to apply critical thinking to the counting sequence. Number-logic riddles progress in difficulty through the counting sequences 1–9, 1–25, 1–50, and 1–100. Sample logic riddles are included in the grades 3–5 activity book (Bamberger and Oberdorf 2010, pages 3–6).

Have students count a set of objects in different ways to help them become more flexible in thinking about part-whole relationships. For example, 52 can be shown as 52 singles, 5 tens and 2 singles, or 3 tens and 22 singles. This also helps students connect counting by ones to counting by groups and singles (tens and ones, for example), so they begin to develop an understanding about place value.

Questions to Ponder

1. What common path games can help students develop their understanding of the one-one counting principle?
2. What additional activities or strategies can you use to help your students become successful counters?

Thinking Addition Means “Join Together” and Subtraction Means “Take Away”

A small argument is brewing in the corner of Mr. Long's first-grade classroom. Four students who have been given a story problem (Carmen has 9 pennies. Brian has 4 pennies. How many more pennies does Carmen have?) have used four different strategies to get three different answers.

Malik says, “The answer is 9 pennies. If Carmen has 9 pennies and Brian has 4 pennies and you take Brian’s 4 pennies away you still have Carmen’s 9 pennies!” (See Figure 1–1.)

Hong rolls her eyes and says, “That’s just plain wrong. The story says ‘more’ so you have to add the 9 pennies and the 4 pennies and that means there are 13 pennies. Thirteen pennies is more than 9 or 4 pennies.”

Figure 1–1 Malik models both sets with pennies.
“But that’s not what the story is saying,” Kendra insists. “See [using counters to illustrate], here are Carmen’s pennies. Here are Brian’s pennies. You want to know how many pennies Brian needs to catch up to Carmen. Look, he needs 5 more pennies.”

“I got the answer of 5 pennies, too. But that’s not how I did it,” Simon tells the group. “Here are Carmen’s pennies [models 9 pennies with counters]. This is a subtracting story, so you have to take away Brian’s 4 pennies. Then there’s 5 pennies left. Nine take away 4 leaves 5.”

### Identifying the Error Patterns

Each student in the above vignette seems to know a strategy for solving a story problem. Three of them use counters to model the problem. Hong has a strategy for figuring out the total of 9 and 4, and knows that in many situations the word *more* cues finding a sum. Simon understands that the problem is suggesting subtraction as the operation with which to solve it, even though his model doesn’t directly match the story. Kendra understands that the story is asking for a comparison between two sets. She also models it and demonstrates the difference between Carmen’s and Brian’s pennies.

Hong has overgeneralized that whenever the word *more* is used in a story problem the operation of addition is required. (Let’s assume she got this idea on her own: when addition problems all use the word *more*, it’s very easy to do.)

Malik’s error is the most interesting. He understands the problem and even models the two sets of pennies. But he’s heard the phrase *take away* too often, so he does this and removes Brian’s 4 pennies. If he had recorded $9 - 4 = \_\_\_$ on paper, he probably wouldn’t have said the answer was 9. With the models in front of him, however, it was easy for him to believe that removing Brian’s 4 pennies still left Carmen’s 9 pennies.

Simon certainly gets the correct answer of 5. But his explanation makes it clear that he has a misconception that with all subtraction there is taking away. Since he believes the story calls for subtraction, the only recourse he has is to take away 4 counters.

### What the Research Says

Carpenter and Moser investigated students’ strategies for solving different types of addition and subtraction story problems and the impact these strategies had on
teachers’ instructional decisions. They determined that young children were able to solve a variety of story problems using strategies that ranged from modeling the problem with counters (including fingers) to recalling basic facts or using derived facts (Carpenter and Moser 1983; Carpenter, Carey, and Kouba 1990). In addition, these researchers identified a hierarchy of problem types—join problems, separate problems, part-part-whole problems, and compare problems—that were more or less complex. For a useful summary of the problem types, see Van de Walle and Lovin (2006, 144).

Below are four story problems that must be solved by subtraction; however, the semantic equation (the equation based on the structure of the story)—and therefore the level of difficulty—is different for each, and “take away” makes sense only for the first:

1.  *Separate: result unknown.*

   Jon had 7 gumballs. He gave 2 to his sister. How many gumballs does he have now?

   Semantic equation: $7 - 2 = ___$

2.  *Join: change unknown.*

   Jon had 2 gumballs. How many more does he need to have 7 altogether?

   Semantic equation: $2 + ___ = 7$


   Jon has some red gumballs and 2 blue gumballs. He has 7 gumballs altogether. How many red gumballs does he have?

   Semantic equation: $___ + 2 = 7$

4.  *Compare: difference unknown.*

   Jon has 7 gumballs. His sister has 2 gumballs. How many more gumballs does Jon have than his sister?

   Semantic equation: $7 - 2 = ___$

---

**Ideas for Instruction**

Prekindergarten and kindergarten teachers can do many things to ensure that students do not enter first grade with misconceptions and overgeneralizations about addition and subtraction. An obvious one, to prevent the misconception that subtraction
means “take away,” is to be sure to use the word *minus* or *subtract* when referring to the subtraction symbol and to reinforce this when students read their equations. Another is to give students many opportunities to take apart a set of objects and put it back together again, perhaps using Cuisenaire® Rods or bicolored counters.

In *Developing Number Concepts Using Unifix Cubes* (1984), Kathy Richardson provides many open-ended problem-based activities that give students a sense that a quantity can be represented in a variety of ways.

Introducing the following activities provides students the opportunity to see that quantities can be represented in many different ways:

- Give prekindergarten or kindergarten students two colors of cubes (the quantity depends on the number being learned) and ask them to represent this number in different ways. If 6 is the number being worked on, children get 12 cubes, 6 of one color and 6 of another color. (All students have the same two colors.) Students also have many blank six-square grids (see Figure 1–2) on which they can place the cubes in various configurations. (They can color in the squares to document each arrangement.) Afterward the students as a class see how many different configurations were created. This activity reinforces the notion of part-part-whole for both addition and subtraction.

![Figure 1–2 Blank Six-Square Grid](image)

- Give prekindergarten or kindergarten students a specific number of cubes or tiles (all the same color) and have them arrange them on a twenty-five-square grid (see Figure 1–3) in various ways. The grid in Figure 1–3 shows one way to make five. Describing it, a student might say, “I put 2 in the top row, 2 in the second row, and 1 in the third row to make my 5.” Another student might describe this same arrangement as, “I put 1 in the first column, 2 in the second column, and 2 in the third column to make 5 altogether.” Sharing ways to make a specific number is an important precursor to later work with addition and subtraction.

- Have prekindergarten students sort a limited set of dominoes (total number of pips per block from zero to six) in order to explore various ways of representing a specific amount. (A full set of double-six dominoes can be used with
kindergartners.) By first grade, students are familiar with different ways to visualize quantities and they can think of part-whole ideas, not just joining and separating relationships. Sample domino problems are included in the grades preK–2 activity book (see Figure 1–4).

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### Using Dominoes to Solve Story Problems

- **Figure 1–3 Grid with Arrangement for the Quantity of Six**

- **Figure 1–4 Using Dominoes to Solve Story Problems**

---

**Name ________________________________ Date __________________**

**Using Dominoes to Solve Story Problems**

- There are 8 pips altogether. Show two different ways to represent this.
- There are 6 pips altogether. Show two different ways to represent this.
- There are 5 pips altogether. Show two different ways to represent this.
- There are 10 pips altogether. Show two different ways to represent this.
Even prekindergarten students can solve simple context-related story problems. For example, if the class keeps track of the date using a calendar, you can ask, “Today is Tuesday, February 3rd. How many more days until it’s the 5th?” Or, “We’ve been in school for 97 days; how many days until the 100th day?”

In either prekindergarten or kindergarten, use a yes/no graph (see Figure 1–5) to help students think about comparative subtraction ideas even before symbolic representations are used. If the question of the day is, “Do you have the letter \(a\) in your first name?” have children put their clothespins on either the yes side or the no side of the graph. Then you can ask, “How many people are on the yes side? How many people are on the no side? How many more people are there on the yes side?”

In first grade, you can incorporate all sorts of appropriate addition and subtraction story problems into your instruction. How Many Snails? (1994), by Paul Giganti, has pictures on each page that students can use to practice counting, naming attributes, determining part-whole relationships, and performing comparative subtraction. Looking at the very first page you could ask, “How many

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**Figure 1–5 Yes/No Graph**
clouds do you see? How many clouds are gray? How many clouds are white? How many clouds are small? How many clouds are large? Think of a number sentence that could be used with this picture.” Make sure the numbers match up with the illustration. When numbers are attached to illustrations, the notion of parts and wholes becomes more real to students. Subtraction stories could also be created that show comparisons: There are 8 clouds in the sky; 4 of them are white and the others are gray. How many gray clouds are there?

In first and second grade, present a variety of problem types and structures to help students move away from the overgeneralization that adding means joining and subtracting means taking away.

Questions to Ponder

1. What manipulatives do you currently have to reinforce the idea of part-whole for addition and subtraction?

2. How might you communicate to families the way you’ll be teaching the concepts of addition and subtraction so that misconceptions and overgeneralizations about addition and subtraction do not occur?

Renaming and Regrouping When Adding and Subtracting Two-Digit Numbers

Mr. Walters has worked hard during the first half of the school year to make sure his second graders understand different ways to represent two-digit numbers. They’ve played games using a hundreds chart and solved story problems using different strategies. Today, as a preassessment, he’s given them the following story problem:

Janet and her dad baked cookies for the class bake sale. On Saturday morning they baked 2 dozen cookies. In the afternoon, they baked 3 dozen cookies. How many cookies will Janet bring to the bake sale?

He’s also provided craft sticks, rubber bands, connecting cubes, a hundreds chart, number lines, and paper and pencils for students to use to help them solve the problems.

Most of the students know that a dozen cookies means 12, and those that don’t ask someone for help. However, their answers include 5 dozen, 50, 60,
Thank you for sampling this resource.

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