Understanding Middle School Math

Cool Problems to Get Students Thinking and Connecting

Arthur Hyde

with Susan Friedlander, Cheryl Heck, and Lynn Pittner

Foreword by Judith Zawojewski

HEINEMANN
Portsmouth, NH
## CONTENTS

**Foreword** ix

**INTRODUCTION** 1

**CHAPTER 1: WHAT YOU TEACH AND HOW YOU TEACH IT** 7

*The Power of KWC: An Alternative to Key Words* 8

*Using KWC to Tap Prior Knowledge* 10

*Using KWC to Structure Group Learning* 12

*Using KWC to Deepen Connections* 13

*Extensions* 16

**CHAPTER 2: SIX BIG IDEAS** 19

*The Research on Mathematical Learning and Teaching* 19

*Principle 1: Engaging Prior Understanding* 19

*Principle 2: The Essential Role of Factual Knowledge and Conceptual Frameworks* 20

*Principle 3: The Importance of Self-Monitoring* 21

*Six Big Ideas: Building on Mathematical Research and Principles* 22

*Big Idea 1: Teachers Broaden Their View of Problem Solving* 22

*Big Idea 2: Making Connections Between the Problem and Their Lives* 34

*Big Idea 3: Creating Multiple Representations of Increasing Abstraction* 43

*Big Idea 4: Students Solving Problems: Same Concept, Multiple Contexts* 51

*Big Idea 5: Cognitively Based Planning for Language, Connections, Contexts, and Representations* 55
Big Idea 6: Integrating Reading Comprehension Strategies and Math Processes via Cognitive Principles 56
Making Meaningful Connections Among Mathematical Concepts 61
The Connectedness of Strands 64
How Does This All Fit Together? 65

CHAPTER 3: NUMBERS AND EARLY ALGEBRA 68
Algebra in the Classroom, Then and Now 68
Partial Products Like You’ve Never Seen Them 69
  Starting Out with Base Ten Blocks and Graph Paper 69
  Moving on to More Abstract Representations and Mental Math 72
Red Dots 74
Algebra Tiles 76
Partial Quotients 80
  Andy’s Inheritance 83
  Square the Digits and Sum the Squares 84
  Summing the Cubes 87
  `The Irrational Tangram 91

CHAPTER 4: PROPORTIONAL REASONING 95
What Proportional Reasoning Looks and Sounds Like in the Classroom 95
  Shampoo Bottle 95
  Cats and Rats 96
  Making Seismometers 99
Developing Students’ Proportional Reasoning Skills 99
  Understanding Differences Between Additive and Multiplicative Transformations 99
  Understanding Ratios 100
  Understanding Rates 105
  Interesting Applications of Rate 110

CHAPTER 5: ALGEBRAIC THINKING AND MODELING 127
Line of Best Fit and Linear Combinations 128
  Positive Slope Situations 128
  Inverse Linear Relations 138
Finite Differences: Quadratic, Cubic, and Beyond 168
  Quadratic Equations 169
  Cubic Equations 176
Conclusion 181
CHAPTER 6: GEOMETRY AND MEASUREMENT 182

Multiple Representations for Solving a Geometry Problem 182
  Ordering Shapes by Two-Dimensional Size 182
  Measuring the Area 191
  Make My Polygon 193
  A Great Extension: Making Dodecagons 196
  What's Your Angle? 198
  Tessellations: A Different Way 202
  Pythagoras 'R' Us 209
  Pythagoras and Similarity 214
  Primitive Pythagorean Triples (PPT) 214

Geometry and the Metric System 216
  Silent Snow, Secret Snow 216

Conclusion 219

CHAPTER 7: DATA ANALYSIS AND PROBABILITY 220

Exploring Experimental Probability 220
  Chevalier de Mere's Game of Chance 220
  Inference and Prediction: Probability Bags 221
  A Plethora of Pigs 225
  Model Building with Montana Red Dog 228

Exploring Possible Outcomes in Theoretical Probability 235
  Combination Pizzas and Permutation Locks 235
  Product Versus Square 242
  Montana Red Dog Follow-Up 245
  De Mere's Bets Follow-Up 246

Concluding Thoughts 246

Appendix 249

References 253

Problem Index 255

Index 259
and vertically draw one of the known sides, in this case, 7. Next, they draw two parallel lines extending out from the top and bottom of the known side. The problem is, we don’t yet know how far to go. But we do know the area (154), so we can figure out the unknown side by repeatedly subtracting chunks that are multiples of 7. Why? Because that way, we’ll always have taken off (or accounted for or covered) one rectangle and still have one rectangle remaining.

Actually, our first move isn’t to have students think about repeated subtraction because that’s generally an unfamiliar topic. Instead, we have students place base ten blocks onto the graph paper and build a rectangle of the given area in order to determine what the unknown side is. (Later we will just use graph paper and no blocks.) Students draw the straight line of the known side and rays for the two unknown sides. Then they shade in the graph squares that correspond to how many squares they have covered up with the base ten blocks.

Andy’s Inheritance

Andy’s Inheritance is a problem from which students learn about regrouping in place value up through the millions. This is a great opportunity for students to incorporate KWC in the solving process.

**FIGURE 3.8**
Andy has inherited one million dollars from his Great-Aunt Edna. He wants his money in cash but he doesn’t want to carry around too many bills in a suitcase or briefcase. He wants a mixture of large and small bills. He decides that he wants some of each of the following **denominations** (the amount of the bill): $100,000, $10,000, $1,000, $100, $10, and $1.

**Question 1: What are several ways Andy can accomplish his goal?**

The first step is to imagine the situation. What’s going on—What do I know for sure?
- Andy inherited $1,000,000.
- Andy wants his money in cash. Andy doesn’t like to have a lot of bills.

Next, what do I want to figure out, find out, or do?
- How can Andy get his million dollars?
- What are some different ways Andy could get his million?

Last, are there any special **conditions**, rules, or tricks I have to watch out for?
- Andy’s money can only be in $100,000 bills, $10,000 bills, $1000 bills, $100 bills, $10 bills, and $1 bills.
- Andy must have some of each denomination.

Try creating a table like the one shown below. Then, starting with the $1 denomination, explain the regrouping you would do to prove this is a million dollars:

<table>
<thead>
<tr>
<th>$100,000</th>
<th>$10,000</th>
<th>$1,000</th>
<th>$100</th>
<th>$10</th>
<th>$1</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>
Question 2: Andy wants some of each of the 6 denominations, but he only wants to carry 100 bills. Can you figure out several ways he can do this?

Question 3: In the table below, there are several examples of ways to make $1,000,000 with these bills. However, some of these examples violate the special conditions that Andy required in question 2. Which examples will not work? Note the new constraint: only use 100 bills.

<table>
<thead>
<tr>
<th>$100,000</th>
<th>$10,000</th>
<th>$1,000</th>
<th>$100</th>
<th>$10</th>
<th>$1</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>18</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>38</td>
<td>18</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>49</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>8</td>
<td>19</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>20</td>
</tr>
</tbody>
</table>

Do you see any patterns in the table above? In the ones column, the number of $1 bills must be a multiple of ten to get a zero in that place for the million. That multiple of ten then would be added to the tens column (the $10 bills). Most of the numbers in the table end in either 8 or 9 because when the column to the right of them groups by tens in order to become a zero, regrouping will occur in that cell to make it a multiple of ten, and so on through the table.

The Andy’s Inheritance problem helps reinforce the concept of place value in our base ten system. Students must group and regroup each of these denominations. There are other denominations of bills that are in circulation (for example, $2, $5, and $20); we ask students to talk about why those bills aren’t included in the problem.

Square the Digits and Sum the Squares

Square the Digits and Sum the Squares is a rather inductive activity I have done with students and teachers alike. I have a collection of pink index cards, at least enough for each student. The cards are blank on one side and have a number that may be one, two, or three digits long on the other. I fan the cards out in my hand face down and ask each student to take one card at random, then I tell them to put the cards away for a moment.
The Ballad of Buttons and Sleeves

I typically have my students do the Ballad of Buttons and Sleeves investigation after they have had practice working with problems that have similar mathematical structures, which not only enables them to think increasingly more abstractly using more abstract representations but also allows them to take strategies—such as shortcuts to completing a table and identifying what it means to find a “common solution” among tables—and apply them in new contexts with ease.

The Ballad of Buttons and Sleeves

A mom-and-pop variety store sells psychedelic dress shirts (either long- or short-sleeved) from the 1960s onto which they sew mood-sensitive buttons (either large or small). Consider:

- Long sleeve shirts require 13 large buttons and 6 small ones. Large buttons are 7 up the front, 2 on each sleeve, and 2 sewn at the bottom for spares. The small buttons are 2 for the button collars, 1 on each sleeve, and 2 spares.
- Short sleeve shirts have only 7 larger buttons up the front and 2 small buttons for the collar.

Mom and Pop have a limited inventory after 40 years: only 235 large buttons and 90 small buttons and fewer than 20 of each kind of shirt. How can they use all of the buttons to make long-sleeved shirts and short-sleeved shirts? Begin with a KWC.

The first thing I had my students do was visualize these crazy shirts. There was a lot of information for them to digest. Before they proceeded to sink their teeth into this problem I wanted them to draw a picture of what they thought the shirts looked like. I did not have the luxury of owning such apparel, so to help my students sketch their pictures, I brought in a few of my husband’s work shirts. See Figure 5.21. My students completed a KWC while reading through the problem in small groups. See Figure 5.22.

FIGURE 5.21
Algebraic Thinking and Modeling

Figure 5.22

Figure 5.23

<table>
<thead>
<tr>
<th>LARGE BUTTONS</th>
<th>235</th>
<th>SMALL BUTTONS</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>LONG SLEEVE SHIRTS</td>
<td>SHORT SLEEVE SHIRTS</td>
<td>LONG SLEEVE SHIRTS</td>
<td>SHORT SLEEVE SHIRTS</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>27</td>
<td>84</td>
</tr>
<tr>
<td>7</td>
<td>42</td>
<td>24</td>
<td>48</td>
</tr>
<tr>
<td>8</td>
<td>56</td>
<td>21</td>
<td>42</td>
</tr>
</tbody>
</table>

\[13x + 7y = 235\]

\[10y + 55z \rightarrow \text{Some solution to both}\]

\[6x + 2y = 90\]
What was most interesting about the Ballad of Buttons and Sleeves problem was not only how extensive my students’ KWCs were but also how students were able to use their prior knowledge of linear modeling, built from prior activities, to complete the table with ease. See Figure 5.23. Finally, I asked students to write a rule for each type of button and identify how it could be a solution to the problem. In our linear modeling activities, my students had exceeded my expectations.

**FINITE DIFFERENCES: QUADRATIC, CUBIC, AND BEYOND**

Finite differences is a powerful technique for analyzing functions to:

1. determine if the functions are based on polynomials; there are many functions that are not polynomial (for example, exponential \( y = 2^x \));
2. determine what degree is involved if the function is polynomial (for example, linear functions are based on first-degree exponents, not higher than 1: \( y = ax + b \); quadratic functions are based on second-degree exponents: \( y = ax^2 + bx + c \); cubic functions are based on third-degree exponents: \( y = ax^3 + bx^2 + cx + d \); and
3. create an equation that models the data using the proper degree, which involves a series of calculations (thanks to hand-held calculators, much of the tedium surrounding the calculations is removed).

Although the topic of finite differences is often found as only a page or two in an Algebra II textbook, to me the technique is a logical follow-up to Chocolate Algebra with students who are strong in eighth-grade algebra. In fact, some of what we did in Chocolate Algebra in the previous section is very similar.

It's important for teachers to assess their students' prior knowledge to determine readiness and understanding for finite differences—we don't want students to merely memorize the procedures. For example, in the Chocolate Algebra problem, when Susan's students had a $5 chocolate bar and a $2 chocolate bar and $40 to spend, they created a table that revealed five points with values that lay on a straight line when graphed. See Figure 5.24.

<table>
<thead>
<tr>
<th></th>
<th>$5</th>
<th>$2</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

**FIGURE 5.24**
Thank you for sampling this resource.

For more information or to purchase, please visit Heinemann by clicking the link below:


Use of this material is solely for individual, noncommercial use and is for informational purposes only.
Thank you for sampling this resource.

For more information or to purchase, please visit Heinemann by clicking the link below:


Use of this material is solely for individual, noncommercial use and is for informational purposes only.