COGNITION-BASED ASSESSMENT & TEACHING

of Addition and Subtraction
Michael T. Battista

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of Addition and Subtraction

Building on Students’ Reasoning

HEINEMANN
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—Michael Battista
Traditional mathematics instruction requires all students to learn a fixed curriculum at the same pace and in the same way. At any point in traditional curricula, instruction assumes that students have already mastered earlier content and, based on that assumption, specifies what and how students should learn next. The sequence of lessons is fixed; there is little flexibility to meet individual students’ learning needs. Although this approach appears to work for the top 20 percent of students, it does not work for the other 80 percent (Battista, 1999, 2001). And even for the top 20 percent of students, the traditional approach is not maximally effective (Battista, 1999, 2001).

For many students, traditional instruction is so distant from their needs that each day they make little or no learning progress and fall farther and farther behind curriculum demands. In contrast, Cognition-Based Assessment (CBA) offers a cognition-based, needs-sensitive framework to support teaching that enables all students to understand, make personal sense of, and become proficient with mathematics.

The CBA approach to teaching mathematics focuses on deep understanding and reasoning, within the context of continually assessing and understanding students’ mathematical thinking and building on that thinking instructionally. Rather than teaching predetermined, fixed content at times when it is inaccessible to many students, the CBA approach focuses on maximizing individual student progress no matter where students are in their personal development. As a result, you can move your students toward reasonable, grade-level learning benchmarks in maximally effective ways. Designed to work with any curriculum, CBA will enable you to better understand and respond to your students’ learning needs and help you choose instructional activities that are best for your students.

There are six books in the CBA project:

- Cognition-Based Assessment and Teaching of Place Value
- Cognition-Based Assessment and Teaching of Addition and Subtraction
- Cognition-Based Assessment and Teaching of Multiplication and Division
- Cognition-Based Assessment and Teaching of Fractions
- Cognition-Based Assessment and Teaching of Geometric Shapes
- Cognition-Based Assessment and Teaching of Geometric Measurement
Any of these books can be used independently, though you may find it helpful to refer to several because the topics covered are interrelated.

**Critical Components of CBA**

The CBA approach emphasizes three key components that support students’ mathematical sense making and proficiency:

- clear, coherent, and organized research-based descriptions of students’ development of meaning for core ideas and reasoning processes in elementary school mathematics;
- assessment tasks that determine how each student is reasoning about these ideas; and
- detailed descriptions of the kinds of instructional activities that will help students at each level of reasoning about these ideas.

More specifically, CBA includes the following essential components.

**Levels of Sophistication in Student Reasoning**

For many mathematical topics, researchers have found that students’ development of mathematical conceptualizations and reasoning can be characterized in terms of “levels of sophistication” (Battista, 2004; Battista and Clements, 1996; Battista et al., 1998; Cobb and Wheatley, 1988; Fuson et al., 1997; Steffe, 1988, 1992; van Hiele, 1986). Chapter 2 describes a framework that characterizes the development of students’ thinking and learning about addition and subtraction in terms of such levels. This framework describes the “cognitive terrain” in which students’ learning trajectories occur, including:

- the levels of sophistication that students pass through in moving from their intuitive ideas and reasoning to a more formal understanding of mathematical concepts;
- cognitive obstacles that students face in learning; and
- fundamental mental processes that underlie concept development and reasoning.

Figure 1 sketches the cognitive terrain that students must ascend to attain understanding of addition and subtraction of whole numbers. This terrain starts with students’ preinstructional reasoning about addition and subtraction, ends with a formal and deep understanding of addition and subtraction, and indicates the cognitive plateaus reached by students along the way. Not pictured in the sketch are sublevels of understanding that may exist at each plateau level. Note that students may travel slightly different trajectories in ascending through this cognitive terrain, and they may end their trajectories at different places depending on the curricula and teaching they experience.
A Note About the Student Work Samples

Chapter 2 includes many examples of students’ work, which are invaluable for understanding and using the levels. All of these examples are important, for they show the rich diversity of student thinking at each level. However, the first time you work through the materials, you may want to read only a few examples for each type of reasoning—just enough examples to comprehend the basic idea of the level. Later, as you use the assessment tasks and instructional activities with your students, you can sharpen your understanding by examining additional examples both in the level descriptions and in the level examples for each assessment task.

Assessment Tasks

The Appendix contains a set of CBA assessment tasks that will enable you to determine your students’ mathematical thinking and precisely locate students’ positions in the cognitive terrain for learning addition and subtraction. These tasks not only assess exactly what students can do, but they also reveal students’ reasoning and underlying mathematical cognitions. The tasks are followed by a description of what each level of reasoning might look like for each assessment task. These descriptions will help you pinpoint your students’ positions in the cognitive terrain of learning.

Using CBA assessment tasks to determine which levels of reasoning students are using will help you pinpoint students’ learning progress, know where students should proceed next in constructing meaning and competence for the idea, and decide which instructional activities will best promote students’ movement to higher levels of reasoning. It can also help guide your questions and responses in classroom discussions and in students’ small-group work. The CBA website at www.heinemann.com/products/E01271.aspx includes additional assessment tasks that you can use to further investigate your students’ understanding of addition and subtraction.
Instructional Suggestions

Chapter 3 provides suggestions for instructional activities that can help students progress to higher levels of reasoning. These activities are designed to meet the needs of students at each CBA level. The instructional suggestions are not meant to be comprehensive treatments of topics. Instead, they are intended to help you understand what kinds of tasks may help students make progress from one level/sublevel to the next higher level/sublevel.

Using the CBA Materials

Determining Students’ Levels of Sophistication

There are several ways that you can use CBA assessment tasks to determine students’ levels of sophistication in reasoning about addition and subtraction.

Individual Interviews

The most accurate way to determine students’ levels of sophistication is to administer the CBA assessment tasks in individual interviews with students. For many students, interviews make describing their thinking much easier; they are perfectly capable of describing their thinking orally but have difficulty doing it in writing. Individual interviews also allow teachers to ask probing questions at just the right time, which can be extremely helpful in revealing students’ thinking. (Beyond assessment purposes, the individual attention that students receive in individual assessment interviews can also provide students with added motivation, engagement, and learning.)

Whole-Class Discussions

In an “embedded assessment” model—in which assessment is embedded within instruction—you can give an assessment task to your whole class as an instructional activity. Each student should have a student sheet with the task on it. Students do all their work on their student sheets and describe in writing how they solve the task. When all the students are finished writing their descriptions of their solution methods, have a class discussion of those methods. For instance, many teachers have a number of individual students present their solutions on an overhead projector or a document-projection device. As students describe their thinking, ask questions that encourage students to provide the detail you need to determine what levels of reasoning they are using. Also, at times, you can revoice or summarize students’ thinking in ways that model good explanations (but be sure that you provide accurate descriptions of what students say instead of formal versions of their reasoning). After

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Introduction

each different student explanation, you can ask how many students used the strategy
described. It is important that you not only have students orally describe their
solution strategies but that you talk about how they can write and represent their
strategies on paper. For instance, after a student has orally described his strategy, ask
the class, “How could you describe this strategy on paper so that I would understand
it without being able to talk to you?”

To see if students’ written explanations accurately describe their solution
strategies, you can ask selected students to come up to your desk and tell you individ-
ually what they did, which you can then compare to what they wrote.

Individual and Small-Group Work

You can also determine the nature of students’ reasoning by circulating around the
room as students are working individually or in small groups on CBA assessment
tasks or instructional activities. Observe student strategies and ask students to
describe what they are doing as they are doing it. Seeing students actually work
on problems often provides more accurate insights into what they are doing and
thinking than merely hearing their explanations of their completed solutions (which
sometimes do not match what they did). Also, as you talk to and observe students
during individual or small-group problem solving, for students who are having
difficulty accurately describing their work, write notes to yourself on students’
papers that tell you what they said and did (these notes should be descriptive, not
evaluative).

The Importance of Questioning

Keep in mind that the more students describe their thinking, the better they will
become at explaining that thinking, especially if you guide them toward providing
increasingly accurate and detailed descriptions of their reasoning. For instance, if a
student says, “I counted,” ask, “How did you count? Count out loud to show me what
you did. How could you write about what you did?”

As a more specific example, consider a student working on the problem, “Mary
has 5 apples and Liz has 4 apples. How many apples do they have altogether?”
Suppose Jim writes “5 + 4 = 9” as his explanation of his strategy. Ask additional
questions.

Teacher: What did you do to figure out that 5 + 4 = 9?

Jim: I counted.

Teacher: How did you count? Count out loud for me.

Jim: 6, 7, 8, 9.

Teacher: Okay, that’s a great way to solve the problem. How could we write that on
your sheet?

Jim: I wrote that I counted.
Teacher: Great. And what else could you write so I know how you counted?

Jim: I don’t know.

Teacher: What numbers did you say when you counted?

Jim: 6, 7, 8, 9.

Teacher: So, you could write these numbers on your sheet.

Listed below are some questions that can be helpful in conducting individual interviews, interacting with students during small-group work, or conducting a classroom discussion of an assessment task.

- That’s interesting; tell me what you did.
- Tell me how you found your answer.
- How did you figure out this problem?
- I’d really like to understand how you’re thinking; can you tell me more about it?
- Why did you do that?
- What were you thinking when you moved these objects?
- Did you check your answer to see whether it is correct? How?
- Explain your drawing to me.
- What do these marks that you made mean?
- What were you thinking when you did this part of the problem?
- What do you mean when you say …?

Monitoring the Development of Students’ Reasoning

The CBA materials are designed to help you assess levels of reasoning, not levels of students. Indeed, a student might use different levels of reasoning on different tasks. For instance, a student might operate at a higher level when using physical materials such as place-value blocks than when she does not have physical materials to support her thinking. Also, a student might operate at different levels on tasks that are familiar to her or that she has practiced as opposed to tasks that are totally new to her. So, rather than attempting to assign a single level to a student, you should analyze a student’s reasoning on several assessment tasks then develop an overall profile of how she is reasoning about the topic. An example of how this is done appears in Chapter 2.

To carefully monitor and even report to parents the development of student reasoning about particular mathematical topics, many teachers keep detailed records of students’ CBA reasoning levels during the school year. To do this, choose several CBA assessment tasks for each major mathematical topic you will cover during the year. Administer these tasks to all of your students either as individual interviews or as written work at several different times during the school year (say, before and after each curriculum unit dealing with the topic). In addition to noting the tasks used and the date, record what levels each student used on the tasks.
Differentiating Instruction to Meet Individual Students’ Learning Needs

You can tailor instruction to meet individual students’ learning needs in several ways.

Individualized Instruction
The most effective way to meet students’ learning needs is to work with them individually using the levels and tasks to precisely assess and guide students’ learning. This approach is an extremely powerful way to maximize an individual student’s learning.

Instruction by CBA Groups
Another effective way of meeting students’ needs is putting students into groups based on their CBA levels of reasoning about a mathematical topic. You can then look to the instructional suggestions for tasks that will be maximally effective for helping the students in each group. For instance, you might have three or four groups in your class, each consisting of students who are reasoning at about the same CBA levels and need the same type of instructional tasks.

Whole-Class Instruction
Another approach that many teachers have used successfully is selecting sets of tasks that all students in a class can benefit from doing. You do this by first determining the different levels of reasoning among students in the class. Then, as you consider possible instructional tasks, ask yourself,

- “How will students at each level of reasoning attempt to do this task?”
- “Can students at different levels of reasoning succeed on the task by using different strategies?” (Avoid tasks that some students will not have any way of completing successfully.)
- “How will students at each level benefit by doing the task?”
- “Will seeing how different students do the task help other students progress to higher levels of thinking because they are ready to hear new ways of reasoning about the task?”

Also, sets of tasks can be sequenced so that initial problems target students using lower levels of reasoning while later tasks target students using higher levels.

Another way to individualize whole-class instruction is to ask different questions to students at different levels as you circulate among students working in small groups. For instance, for students who are operating on numbers as collections of ones, you might ask if there is another way to count to solve the problem—can they use skip-counting? On the same problem, for students who are already skip-counting, you might ask if they can do the problem without counting (say, by using
number properties and derived facts). Knowledge of CBA levels is invaluable in devising good questions and in asking appropriate questions for different students. In fact, when preparing to teach a lesson, many teachers use levels-of-sophistication descriptions to think about the kinds of questions they will ask students who are functioning at different levels.

Choosing which students to put into small groups for whole-class inquiry-based instruction is also important. If you think of your students’ CBA levels of reasoning on a particular type of task as being divided into three groups, you might put students in the high and middle groups together or students in the middle and low groups together. Generally, putting students in the high and low groups together is not effective because their thinking is likely to be too different.

**Assessment and Accountability**

As a consequence of state and federal testing and accountability initiatives, most school districts and teachers are looking for materials and methods that will help them achieve state performance benchmarks. CBA is a powerful tool that can help you help your students achieve these benchmarks by:

- monitoring students’ development of reasoning about core mathematical ideas;
- identifying students who are having difficulties learning these ideas and diagnosing the nature of these difficulties;
- understanding the nature of weaknesses identified by annual state mathematics assessment results along with causes for these weaknesses; and
- understanding a framework for remediating student difficulties in conceptually and cognitively sound ways.

**Moving Beyond Deficit Models**

The CBA materials can help you move beyond the “deficit” model of traditional diagnosis and remediation. In the deficit model, teachers wait until students fail before attempting to diagnose and remediate their learning problems. CBA offers a more powerful, preventive model for helping students. By using CBA materials to appropriately pretest students on core ideas that are needed for upcoming instructional units, you can identify which students need help and the nature of the help they need before they fail. By then using appropriate instructional activities, you can help students acquire the core knowledge needed to be successful in the upcoming units—making that instruction effective rather than ineffective for these students.

**The Research Base**

Not only have these materials gone through extensive field testing with both students and teachers, but the CBA approach is also consistent with major scientific theories describing how students learn mathematics with understanding. These theories agree
that mathematical ideas must be personally constructed by students as they intentionally try to make sense of situations. Furthermore, to be effective, mathematics teaching must carefully guide and support students’ construction of personally meaningful mathematical ideas (Baroody and Ginsburg, 1990; Battista, 1999, 2001; Bransford, Brown, and Cocking, 1999; De Corte, Greer, and Verschaffel, 1996; Greeno, Collins, and Resnick, 1996; Hiebert and Carpenter, 1992; Lester, 1994; National Research Council, 1989; Prawat, 1999; Romberg, 1992; Schoenfeld, 1994; Steffe and Kieren, 1994; von Glasersfeld, 1995). Research shows that when students’ current ideas and beliefs are ignored, their development of mathematical understanding suffers. And conversely, “There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students’ changing conceptions as instruction proceeds” (Bransford et al., 1999, p. 11).

The CBA approach is also consistent with research on mathematics teaching. For instance, based on their research in the Cognitively Guided Instruction program, Carpenter and Fennema concluded that teachers must “have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes” (1991, p. 11). Indeed, a number of studies have shown that when teachers learn about such research on students’ mathematical thinking, they can use that knowledge in ways that positively impact their students’ mathematics learning (Carpenter et al., 1998; Cobb et al., 1991; Fennema and Franke, 1992; Fennema et al., 1996; Steff and D’Ambrosio, 1995). These materials will enable you to:

- develop a detailed understanding of your students’ current reasoning about specific mathematical topics and
- choose learning goals and instructional activities to help your students build on their current ways of reasoning.

Indeed, these materials provide the kind of coherent, detailed, and well-organized research-based knowledge about students’ mathematical thinking that research has indicated is important for teaching (Fennema and Franke, 1992).

Research also shows that using formative assessment can produce significant learning gains in all students (Black and Wiliam, 1998). Furthermore, formative assessment can be especially helpful for struggling students, so it can reduce achievement gaps in mathematics learning. The CBA materials offer teachers a powerful type of formative assessment that monitors students’ learning in ways that enable teaching to be adapted to meet students’ learning needs. “For assessment to function formatively, the results have to be used to adjust teaching and learning” (Black and Wiliam, 1998, p. 142). To implement high-quality formative assessment, the major question that must be asked is, “Do I really know enough about the understanding of my pupils to be able to help each of them?” (Black and Wiliam, 1998, p. 143). CBA materials help answer this question.
Using CBA Materials for RTI

Response to Intervention (RTI) is a school-based, tiered prevention and intervention model for helping all students learn mathematics. Tier 1 focuses on high-quality classroom instruction for all students. Tier 2 focuses on supplemental, differentiated instruction to address particular needs of students within the classroom context. Tier 3 focuses on intensive individualized instruction for students who are not making adequate progress in Tiers 1 and 2.

CBA can be effectively used for all three RTI tiers. For Tier 1, CBA materials provide extensive, research-based descriptions of the development of students' learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, CBA descriptions enable you to better understand and monitor each student's mathematics learning through observation, embedded assessment, questioning, informal assessment during small-group work, and formal assessment. You can then choose instructional activities that meet your students' learning needs—whole-class tasks that benefit students at all levels; different tasks for small groups of students at the same levels; individualized supplementary student work. For Tier 3, CBA assessments and level-specific instructional suggestions provide road maps and directions for giving struggling students the long-term individualized instruction sequences they need.

Supporting Students’ Development of Mathematical Reasoning

CBA materials are designed to help students move to higher levels of reasoning. It is important, however, that instruction not demand that students “move up” the levels with insufficient cognitive support. Such demands result in students rote memorizing procedures that they cannot make personal sense of. Jumps in levels are made internally by students, not by teachers or the curriculum. This does not mean that students must progress through the levels with no help. Teaching helps students by providing them with the right kinds of encouragement, support, and challenges. Good teaching has students work on problems that stretch, but do not overwhelm, their reasoning; asks good questions; has students discuss their ideas with other students; and sometimes shows students ideas that they don’t invent themselves. But when we show students ideas, we should not demand that they use them. Instead, we should try to get students to adopt new ideas because students make personal sense of the ideas and see the new ideas as better than the ideas they currently possess.
Chapter 1

Introduction to Understanding Addition and Subtraction

There are two major components in the development of students’ understanding of and proficiency with addition and subtraction of whole numbers. First, students must develop an understanding of the meanings of the operations of addition and subtraction, including recognizing when each operation is appropriate. Initially, this understanding is based on students’ understanding of counting and the meaning of numbers. Second, students must develop an understanding of and proficiency with methods for adding and subtracting numbers utilizing a variety of reasoning strategies and culminating in understanding paper-and-pencil computational algorithms.

Understanding Counting and Numbers

To understand adding and subtracting numbers, students must first understand numbers as telling how many objects are in sets, which they determine primarily by counting. To correctly count a set of objects, students must properly associate a fixed sequence of counting words (one, two, three, and so on) with an organized sequence of “tagging” (touching, pointing, marking) the elements in the set in such a way that each and every object is associated with one and only one counting word.

There are several additional ways that students might determine the number of objects in a set. For small sets, students often use subitizing, which means that they instantly recognize the number of objects in the set. Most students can subitize sets of 2, 3, and 4 objects—or somewhat larger sets if the objects are arranged in a familiar pattern, like the 6 dots on a die, or 5 fingers in the following open hand finger pattern.
Students who are more advanced in place-value reasoning might recognize, without counting, simple place-value block representations of numbers. For instance, a student might almost instantly, and without counting, recognize that the configuration below represents 35. (See *Cognition-Based Assessment and Teaching of Place Value* for more discussion of the critically important topic of place value.)

Understanding Adding and Subtracting Situations

Once students understand numbers, they can progress to developing conceptual understanding of physical situations that give rise to addition and subtraction. There are three basic situations (see Carpenter et al., 1999, for a more detailed description).

Addition

Initially, students understand addition in terms of physically joining sets of objects: “Jon has 5 cookies. His mother gives him 3 more cookies. How many cookies does he have now?” A somewhat more abstract addition situation occurs when two sets of objects are mentally rather than physically joined: “Jon has 5 red marbles and 3 green marbles. How many marbles does he have altogether?” (Note that students might solve the second problem by physically joining the two sets of marbles.) In the first problem, there is a physical act of joining (Jon’s mother gives him 3 cookies); in the second, Jon has both sets already, and there is no physical joining action described in the problem.
Subtraction

Initially, students understand subtraction in terms of physically taking away one set of objects from another: “Jon has 8 cookies. He gives 3 cookies to his mother. How many cookies does he have now?”

Comparison

Students also use addition and subtraction to solve comparison problems such as, “Jonathan has 8 black marbles and Emily has 5 gray marbles. How many more marbles does Jonathan have?” However, the most basic way to solve this problem uses the fundamentally important mathematical process of setting up a one-to-one correspondence between the gray marbles and a matching subset of black marbles.

Even though setting up a one-to-one correspondence is critically important mathematically, most students are taught to solve this type of problem using addition or subtraction. To use subtraction and separating, we can reason: “If I take 5 marbles away from 8, there are 3 marbles left.” Thinking in terms of a missing addend addition problem, we can reason: “How many more marbles do I need to join to Emily’s 5 to get 8? I need 3.”

In the initial situations for both addition and subtraction, the goal is to find the result of physically operating (joining or separating) the two given numbers. Symbolically, this problem is represented by $A + B = x$ or $A - B = x$. Other situations that can be modeled with addition and subtraction occur when one of the first two numbers in addition or subtraction is unknown, such as $A + x = C$ or $x - B = C$.

Relating Addition and Subtraction

It is extremely important for students to develop an understanding of the inverse relationship between addition and subtraction. Algebraically, this relationship is expressed by the statement:

$$a - b = c \text{ if and only if } b + c = a$$

Physically, this relationship can be understood as saying that if I remove $b$ objects from a set of $a$ objects, I will return to having $a$ objects if I add $b$ objects back to the set.
Understanding Students’ Levels of Sophistication for Addition and Subtraction

The CBA levels provide a detailed description of the development of students’ reasoning about addition and subtraction of whole numbers. This detail is critical for tailoring instruction to meet students’ learning needs. However, when you are first learning a set of CBA levels, the amount of detail can be overwhelming. So, keep in mind that understanding CBA levels comes in stages and develops over time. First, you will learn the major features of the levels-of-sophistication framework for addition and subtraction. As you use CBA with your students, you will learn the details of the framework.

Zooming Out to Get an Overview

To begin understanding the CBA levels for addition and subtraction, it is important to develop an understanding of the overall organization of the levels. The chart below shows the CBA addition and subtraction levels in a “zoomed-out” view.

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Knowledge and Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
<td>Student does not understand addition and subtraction situations.</td>
<td>No use of algorithms</td>
</tr>
<tr>
<td>Level 1</td>
<td>Student adds or subtracts numbers as collections of ones (no skip-counting by place value).</td>
<td>No use or rote use of algorithms</td>
</tr>
<tr>
<td>Level 2</td>
<td>Student adds or subtracts numbers by skip-counting place-value parts.</td>
<td>No use or use of algorithms with weak or no connection between place value and algorithms</td>
</tr>
<tr>
<td>Level 3</td>
<td>Student adds or subtracts numbers by combining or separating place-value parts.</td>
<td>Explicit use of place value in informal multidigit arithmetic; emerging but incomplete understanding of place value in algorithms</td>
</tr>
<tr>
<td>Level 4</td>
<td>Student uses and understands expanded addition and subtraction algorithms.</td>
<td>Place-value understanding of expanded algorithms (through hundreds)</td>
</tr>
<tr>
<td>Level 5</td>
<td>Student uses and understands traditional addition and subtraction algorithms.</td>
<td>Place-value understanding of traditional algorithms (through hundreds)</td>
</tr>
</tbody>
</table>
These broad levels describe the major ways that students think about adding and subtracting numbers. The levels suggest an overall learning sequence: students learn about adding and subtracting single-digit numbers first, then about adding and subtracting multidigit numbers, and then about algorithms for adding and subtracting. The zoomed-out view makes clear that if students learn algorithms before developing underlying concepts, their learning will be rote.

There is also an important progression indicated by the rows of the table. At the top row, Level 0, students do not understand the concepts of addition and subtraction. At the bottom two rows, Levels 4 and 5, students have developed a sound understanding of and fluency with algorithms for addition and subtraction, first with expanded algorithms then with traditional algorithms. Levels 1–3 describe how students progress from initial counting-by-ones understanding of addition and subtraction, to counting-by-tens strategies, to sophisticated, property-based, noncounting procedures that prepare them for deep conceptual understanding of computational algorithms.

In summary, as you examine the levels of sophistication in students’ reasoning, you will observe the following general progression.

1. Students’ beginning strategies for adding and subtracting whole numbers use physical materials to model the action or relations described in problems, and students use counting to determine how many elements are in sets of objects.

2. From physical modeling, students progress to counting-only strategies; first students count by ones, then by place-value parts (tens, hundreds, and so on).

3. Students then combine and decompose numbers by place-value parts without counting, using known facts or deriving new facts from known facts. For example, students decompose 34 + 45 into tens and ones, saying, “30 plus 40 equals 70, 4 plus 5 equals 9, and 70 plus 9 is 79.”

4. Finally, students understand and use symbolic algorithms. Importantly, students’ understanding of algorithms depends critically on their understanding of place value and other properties of numbers such as the commutative property of addition.

Understanding Algorithms

A computational algorithm is a precisely specified sequence of actions performed on written symbols that systematically solves one general type of computation problem. The levels of sophistication in Cognition-Based Assessment (CBA) describe students’ development of core concepts and ways of reasoning about addition and subtraction. An important part of this development is understanding and becoming fluent with using computational algorithms. However, if algorithms are taught too early in students’ development of reasoning about addition and subtraction, students cannot understand the algorithms conceptually, so they learn them by rote. Indeed, most students in traditional instruction learn traditional algorithms for addition and subtraction by rote without
understanding the underlying number properties. Chapter 2 contains a special section on understanding and determining levels of sophistication in students’ use of computational algorithms.

**Zooming In to Meet Individual Students’ Needs**

Understanding individual students’ reasoning precisely enough to maximize their learning or remediate a learning difficulty requires a more detailed picture. We must “zoom in” to see sublevels (see Figure 1.1). The “jumps” between sublevels must be small enough that students can achieve them with small amounts of instruction in relatively short periods of time.

Imagine students trying to climb the plateaus in the cognitive terrain described by CBA levels. In situation A, the student has to make a cognitive jump that is too great. In situation B, the student can get from Level 1 to Level 2 by using accessible sublevels as stepping-stones. To provide students with the instructional guidance and cognitive support they need to develop a thorough understanding of mathematical ideas, you need to understand and use the sublevels. Chapter 2 provides detailed descriptions and illustrations of all the CBA levels and sublevels for addition and subtraction.

![Figure 1.1 Accessible Cognitive Jumps](image-url)
Notes

1. To simplify the descriptions in this document, we often discuss only ones and tens in two-digit numbers. Similar, but more complicated, ideas occur for numbers containing more than two digits.

2. Implementing reasoning strictly verbally is more sophisticated than implementing it concretely or pictorially. Consequently, when investigating students’ CBA levels, we should always determine and note if students need visible or physical material to implement their reasoning. So, for instance, if a CBA assessment task does not provide visual/physical material and students ask for it, ask them if they can do the problem without the material. Then let them check their answers with the material to see whether their answers are correct or not.

3. At certain times in students’ learning, it can be helpful for them to use place-value blocks. But be careful about how you and students verbally refer to these blocks—the numerical value, not the shape, should be prominent. For instance, refer to the blocks below as follows.

<table>
<thead>
<tr>
<th>“one” or “one-block”</th>
<th>“ten” or “ten-block”</th>
<th>“hundred” or “hundred-block”</th>
</tr>
</thead>
<tbody>
<tr>
<td>not “cube”</td>
<td>not “strip” or “long”</td>
<td>not “flat”</td>
</tr>
</tbody>
</table>

Also, the goal in using place-value blocks is to help students develop reasoning about numbers, not blocks. For instance, having students learn that $23 + 45$ can be found by joining the place-value block representation of each number, without understanding specifically how this representation is related to the manipulations of numbers, is unlikely to be productive for students. In fact, it’s important that when representing, say, 23 with place-value blocks, students recognize that the numeral 2 and the 2 ten-blocks both represent 2 tens or 20.
Thank you for sampling this resource.

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