Fostering Geometric Thinking

A Guide for Teachers, Grades 5–10

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Introduction

In 1982, Sir Michael Atiyah addressed a group of mathematicians on the topic "What is geometry?" Atiyah—renowned Oxford mathematician, Fields Medalist, and winner of the 2004 Abel Prize—offered the view that effective mathematical problem solving depends on complementary ways of thinking:

> Broadly speaking I want to suggest that geometry is that part of mathematics in which visual thought is dominant whereas algebra is that part in which sequential thought is dominant. This dichotomy is perhaps better conveyed by the words *insight* versus *rigour* and both play an essential role in real mathematical problems.
>
> The educational implications of this are clear. We should aim to cultivate and develop both modes of thought. It is a mistake to overemphasise one at the expense of the other and I suspect that geometry has been suffering in recent years. (Atiyah 2003, 29)

The following may illustrate his point about the two complementary modes of thought.

→ The straight line $y = (7/12)x + 1/4$ passes through two points with integer coordinates (3, 2) and (–9, –5). Are there other points on this straight line with both coordinates integers?

One way to think about the problem has a sequential feel to it and is grounded in attention to the numerical relationship between a number x and the number $(7/12)x + 1/4$:

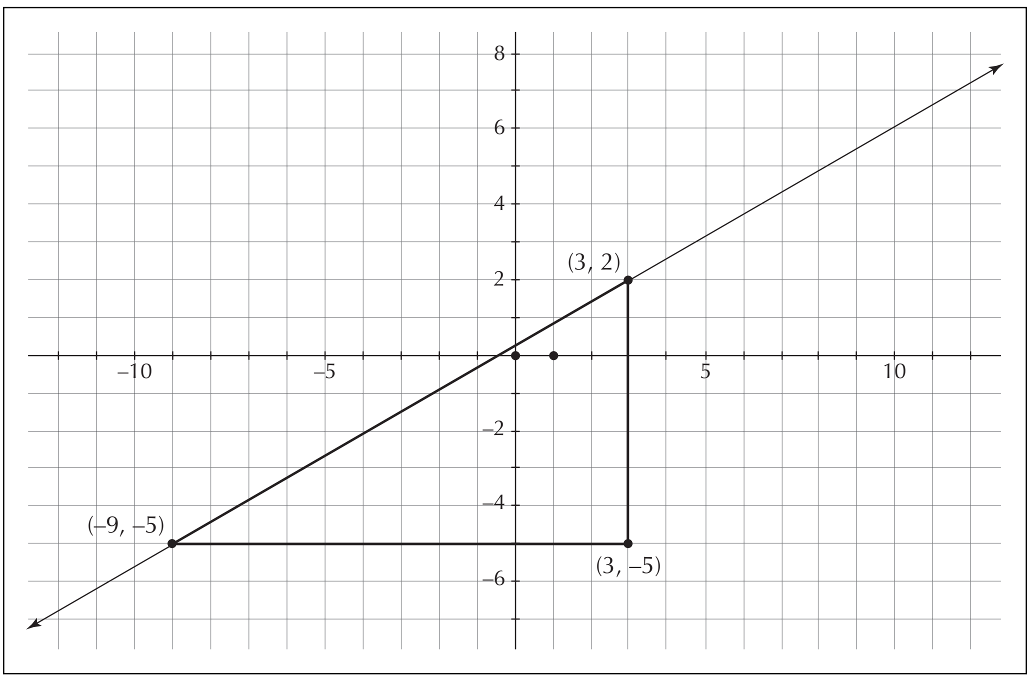
- Saying $(7/12)x + 1/4$ is an integer, with x also an integer, implies that $7x/12 + 1/4$ is an integer, which in turn implies that
- $7x + 3$ is divisible by 12, so
- $7x$ will leave a remainder of 9 when divided by 12
- So, $(x, (7/12)x + 1/4)$ works if and only if x is of the form $3 + 12m$ for some integer m. And this implies that there are

- an infinite number of points on the line with integer coordinates. For example, the two given points, (3, 2) and (–9, –5), correspond to $m = 0$ and $m = -1$, respectively, and this invites thinking about $m = 1$, which yields the next point on the line with integer coordinates, (15, 9).

Another way of thinking about the problem, which seems more visual in character, grows from attending to the role that similar triangles play in the graphs of linear functions. For example, there are infinitely many right triangles, with vertex (–9, –5) and hypotenuse on the line through (–9, –5) and (3, 2), similar to the right triangle with vertices (–9, –5), (3, 2), and (3, –5). To construct such similar triangles, one could double the lengths of the sides of that right triangle, triple them, quadruple them, and so on from which one can see that there are an infinite number of points with integer coordinates on the line (see Figure I–1).

We want students to learn to think in both kinds of ways, so Atiyah's final sentence about "geometry suffering" expresses a sentiment we share, and one that was a strong motivator for writing this book. We believe that U.S. students have been given too little exposure to geometry and geometrical thinking, particularly in the middle grades. We also believe this negligence has had detrimental effects, as recent data show.

FIGURE I–1



In a summary of research on teaching and learning geometry at the K–12 level, Clements concluded that "U.S. curriculum and teaching in the domain of geometry is generally weak, leading to unacceptably low levels of achievement" (2003, 152). Geometry has been characterized as "the forgotten strand" (Lappan 1999) among those identified in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000).

Beyond shape recognition, geometric concepts have received scant attention in many curriculum materials for grades K–8 (AAAS 2000; NCTM 2000). Teachers may not even teach what little geometry can be found in those materials (Clements 2003; Porter 1989; Fuys, Geddes, and Tischler 1988). Even with curriculum materials that provide opportunity for studying deeper concepts in geometry, teachers may lack knowledge and skills to teach geometry effectively (Goldsmith, Mark, and Kantrov 1998).

The need for improvement in geometry teaching and learning in the middle and high school grades is clearly evident in international comparisons such as Trends in International Mathematics and Science Study (TIMSS), which focuses on grades 4, 8, and 12, and Programme for International Student Assessment (PISA), which compares the performance of 15-year-old students. Consistently over recent years, in TIMSS for grade 8, geometry and measurement are the areas of weakest performances for U.S. students (Ginsburg et al. 2005; Mullis et al. 2001; Beaton et al. 1997). By the end of high school, the scores of U.S. students were near the bottom in the TIMSS study of advanced mathematics, with U.S. geometry performance the lowest of all participating nations (Mullis et al. 1998).

In an analysis of the 2003 TIMSS and PISA assessments, Ginsburg et al. (2005) compared the results in the countries that participated in TIMSS at grades 4 and 8 and in PISA.[1] They concluded that the United States spends 50 percent less time on geometry in grade 8 than the other countries. They concluded also that measurement and geometry were the clear weaknesses: U.S. students' performance on measurement items was statistically lower than their overall score on TIMSS-4 and TIMSS-8; their performance on geometry items was statistically lower than their overall performance on TIMSS-8 and PISA.

There may be a glimmer of hope in all these worrisome data. The same TIMSS–PISA analysis determined that U.S. students' performance on statistics items (e.g., data, probability, uncertainty) was statistically higher than their overall score on TIMSS-4, TIMSS-8, and PISA. Perhaps not by coincidence, it was also determined that the United States spends 50 percent more time on data and statistics at grade 4 than the other countries.

[1]The twelve countries that participated in TIMSS-4, TIMSS-8, and PISA are Australia, Belgium, Hong Kong, Hungary, Italy, Japan, Latvia, the Netherlands, New Zealand, Norway, the Russian Federation, and the United States.

All these data provide compelling evidence that:

- We don't teach enough geometry or geometric thinking in the middle grades.
- The effects of this negligence show up in international assessments.
- The opposite is true for statistics: greater relative emphasis in instruction, arguably leads to greater relative results in international comparisons.

The conclusion seems clear: As a country, we need to increase attention to geometry in the middle grades. Further, what students learn should be rich in geometric reasoning.

A recent analysis demonstrated that geometric skills on the TIMSS assessments are definitely related to students' competence with higher-order mathematics processes including: logical reasoning, applications of knowledge in arithmetic and geometry, management of data and procedures, and proportional reasoning. This finding has led to the suggestion that teaching geometric thinking in the middle grades is at least as important as teaching algebraic thinking (Tatsuoka, Corter, and Tatsuoka 2004).

When it comes to deciding how to enrich middle-grades geometry learning, there are sources of guidance to turn to. For example, the Grades 6–8 Geometry and Measurement Standards of the National Council of Teachers of Mathematics (NCTM) use the following phrases to describe learning expectations (NCTM 2000, 232, 240) that portray a very rich diet of geometric learning for students.

- "Analyze characteristics and properties of . . . geometric shapes and develop mathematical arguments about geometric relationships"
- "Describe spatial relationships"
- "Apply transformations"
- "Use visualization, spatial reasoning, and geometric modeling to solve problems"
- "Understand measurable attributes of objects"

Further, in 2006, NCTM complemented the Principles and Standards by describing compact sets of Curriculum Focal Points in each grade through grade 8 (www.nctmmedia.org/cfm). Among the focal points are:

- Grade 5: "Describing three-dimensional shapes and analyzing their properties, including volume and surface area."

- Grade 7: "Understanding and applying proportionality, including similarity."
- Grade 8: "Understanding two- and three-dimensional space and figures using distance and angle."

A central purpose of this book is in its title: to enhance the chances of teachers fostering geometric thinking in their classrooms so that students will learn to use geometric thinking as a complement to algebraic thinking in problem solving. Teachers' understanding of geometric thinking, along with their ability to help students understand and employ geometric thinking, seems like a very important piece of school mathematics.

In its 2005 book, *How Students Learn Mathematics in the Classroom*, the National Research Council offered three core principles for success in mathematics instruction: (1) engaging prior understandings, (2) organizing knowledge around core concepts, and (3) supporting metacognition. It is our view that metacognition, particularly in the area of geometry, is hardly visible in school mathematics. Fostering students' attention to their geometric thinking to help them think more productively when solving problems is very important.

We believe, however, that a prerequisite to teachers accomplishing this goal is their own understanding of geometric thinking. Those two goals for teachers—*understanding geometric thinking* and *fostering geometric thinking*—are at the heart of this book. The following features are intended to help in this regard.

- *Chapter 1, Geometric Habits of Mind*, contains a description of productive geometric thinking in terms of four geometric habits.

An organization around three clusters of topics that echo the NCTM recommendations follows:

- *Chapter 2, Geometric Relationships:* Just as important in problem solving as algebraic thinking about numerical relationships is geometric thinking about relationships within and between geometric figures. (Consider, for example, the relationship between each of the similar triangles considered in solving the problem about finding points on the straight line with both coordinates integers.)
- *Chapter 3, Geometric Transformations*, with emphasis on their effect on geometric objects, particularly invariance effects. For example, does a particular transformation preserve length-of-line segments? Does it preserve area?
- *Chapter 4, Geometric Measurement*, considers measuring such as the length, area, angle, size, and volume.

A summary of recommendations for instructional practice wraps up this book:

- *Chapter 5, Principles for Fostering Geometric Thinking,* makes principles that have been implicit in previous chapters explicit: a steady diet of geometric problem solving is valuable for middle graders; focusing on communication in middle-grades geometry is important; and middle-grades geometry can and should form the groundwork for high school geometry.

Accompanying this book is a DVD containing images of students solving geometry problems. At certain points throughout the book, readers are referred to relevant video clips on the DVD. For the most part, these images, we believe, exemplify indicators of the geometric habits of mind. In addition, there are images that pertain to the importance in geometric problem solving of teacher questioning and of mathematical language and communication.