INTRODUCTION

This book is written for teachers who want to reflect on, and perhaps redirect, their thinking about the learning and teaching of prealgebra and algebra, in light of the movement to base curriculum, instruction, and assessment on clearly defined standards. By focusing on explicit criteria for what is important to learn, the K through 12 standards movement has shifted the attention of educators from teaching toward learning. In this book we try to follow that shift by emphasizing learning issues around mathematical thinking as much as strategies for teaching mathematical thinking.

The book's ideas are derived in good measure from three teacher-enhancement projects in which we at the Education Development Center (EDC) have been involved[1]: the Linked Learning in Mathematics project, the Leadership for Urban Mathematics Reform (LUMR) project, and the Assessment Communities of Teachers (ACT) project. A central belief underlying our work has been that attention to learning, balanced with attention to teaching, is necessary in mathematics teachers' professional development. This is different from the norm. Typically, even when staff development focuses on standards-based mathematics education, the thrust of teachers' work together has been to translate what they hear and experience into instructional terms. In part, this phenomenon is due to the dominance that "how-to" considerations have had in teachers' professional development over considerations about how learning happens.

In our work with teachers, our goal has been to provide something different: experiences that prompt deep and personal reflections about learning, and that cause teachers to bring to the surface the mindsets that underlie their theories about learning that guide their own learning and, eventually, color the way they teach. As a result, the approaches to professional development emphasize productive habits of thinking, such as those that are associated with the effective understanding and use of algebra. In addition, we have emphasized the analysis of student work, using habits of thinking as criteria for analysis. Discussions are structured to invite teachers to reflect on their own thinking, as well as that of colleagues and the students whose work they analyze. They are structured to measure the gaps between what characterizes productive algebraic thinking and the think-

ing patterns that typically characterize students' development of algebraic thinking, as reflected in student work.

We believe that grounding professional development in the understanding of productive habits of thinking

- gives substance to efforts at cross-grades articulation. In fact, the Linked Learning in Mathematics and LUMR projects have worked with mixed groups of middle-grades and high school teachers, often analyzing cross-grade student work on the same mathematics task.
- creates opportunities for action toward equity in schools, in the sense that student efficacy can be grounded in teachers' knowledge of how to foster the development of productive habits of algebraic thinking. With its emphasis on student thinking and learning in the analysis of student work, as opposed to emphasizing only whether the students "got it," fresh insights are often gained about what student work reveals about thinking patterns. In turn, the focus on threads of productive mathematical thinking as they develop across grades makes it possible for teachers and students to become agents in this development.

So much of the work on understanding habits of thinking would have limited appeal and value to teachers if it didn't lead eventually to decisions about classroom practice. It is one thing to see threads of algebraic thinking in a student response or to notice gaps in thinking; it is quite another thing to know how to teach the student to develop his or her thinking in productive ways. Much of our project work with teachers has included discussions, often based on analysis of student work, about the gap between effective mathematical thinking and what students reveal about their thinking. We have noticed a set of critical issues that seem to arise naturally in these discussions. These issues, emerging typically from the inferences teachers make about the way a particular student approached a problem, become the bridges to talking about classroom practice, and form the organizational framework of this book.

The first chapter sets the stage for the remaining chapters by describing our perspective on algebraic thinking and by providing a framework for the use of classroom questions to foster the development of algebraic thinking in students from grades 6 through 10. Each subsequent chapter uses the following elements to address its particular focus issue:

- An account, from the learner's perspective, of why the issue is important and how it connects to our algebraic-thinking framework. Whenever possible and appropriate, the accounts are supported by references to learning research. Also, on occasion, we have been able to support the descriptions with samples of student work gathered in our projects.
- An account, from the teacher's perspective, of some key classroom considerations in order to support students' growth of algebraic thinking. Particular attention is given to the kinds of questions teachers can ask students.

Chapter 1	Besides serving the purpose of orienting the reader to our approach to algebraic thinking through three algebraic habits of mind, this chapter addresses the role of teacher questioning in the fostering of algebraic thinking. In particular, the questions: *What kinds of open questions can foster algebraic thinking, and when is it best to use them?*
Chapter 2	One traditional definition of *algebra* is "generalized arithmetic." This points to the need for careful bridging between arithmetic and algebra. *What can be done to help students build on arithmetic computational skills to develop their algebraic thinking?*
Chapter 3	Similarly, there are mathematical experiences that students have in early grades that are not particularly arithmetical or computational, like those related to number sense. *What can be done to help students build on number sense to develop their algebraic thinking?*
Chapter 4	Any effective bridging from elementary school mathematics to algebra will involve generalization about the operations being used in arithmetic. *What can be done to foster operation sense or the capacity to generalize about number-system structure?*
Chapter 5	Similarly, as students engage with functions and relations in prealgebra and algebra, they need to know how to generalize in that domain. *What can be done to foster the capacity to generalize about functional relations?*
Chapter 6	Symbolic representation and manipulation are the lifeblood of algebra. *How and when should students be expected to engage in symbolic representation and manipulation in algebra?*
Chapter 7	Really productive algebraic thinkers can let their thinking flow easily among different algebraic representations. *How can students be helped to understand, use, and link multiple representations?*

- A few brief stories from our projects. We have found that a critical element in teachers informing their teaching with a greater understanding of learning has been their willingness to explore their own thinking. Therefore, whenever possible, we have incorporated brief stories from the projects about teachers' explorations into learning. These stories are not intended to describe the projects from which they originate, as much as they are intended to convey some of what was learned about algebraic thinking, through the teachers' mathematical explorations and discussions.
- A set of mathematics activities that might prompt similar reflections by readers about their own algebraic thinking, for possible use with their students. Each activity is accompanied by a discussion of its solution, pointers to possible extensions, and connections to the algebraic-thinking framework. Many of the activities are suitable for students across the 6 through 10 grade span.
- A brief list of references for further reading

Note

1. All three are or were National Science Foundation teacher-enhancement projects: 1. Linked Learning in Mathematics: Marquette University (1997–),

ESI-9619366, involves Milwaukee teachers; 2. Leadership for Urban Mathematics Reform (LUMR): Education Development Center (1994-1997), ESI-9353449, involved teachers in Durham, Los Angeles, Milwaukee, St. Louis, San Diego, and Worcester; 3. Assessment Communities of Teachers (ACT): Pittsburgh Public Schools (1994–1997), ESI-9353622, involved teachers in Dayton, Memphis, Milwaukee, Pittsburgh, San Diego, and San Francisco.