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COGNITION-BASED ASSESSMENT & TEACHING

of Fractions

Building on Students’ Reasoning

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—Michael Battista
Introduction

Traditional mathematics instruction requires all students to learn a fixed curriculum at the same pace and in the same way. At any point in traditional curricula, instruction assumes that students have already mastered earlier content and, based on that assumption, specifies what and how students should learn next. The sequence of lessons is fixed; there is little flexibility to meet individual students’ learning needs. Although this approach appears to work for the top 20 percent of students, it does not work for the other 80 percent (Battista 1999, 2001). And even for the top 20 percent of students, the traditional approach is not maximally effective (Battista 1999, 2001). For many students, traditional instruction is so distant from their needs that each day they make little or no learning progress and fall farther and farther behind curriculum demands. In contrast, Cognition-Based Assessment (CBA) offers a cognition-based, needs-sensitive framework to support teaching that enables all students to understand, make personal sense of, and become proficient with mathematics.

The CBA approach to teaching mathematics focuses on deep understanding and reasoning, within the context of continually assessing and understanding students’ mathematical thinking, then building on that thinking instructionally. Rather than teaching predetermined, fixed content at times when it is inaccessible to many students, the CBA approach focuses on maximizing individual student progress, no matter where students are in their personal development. As a result, you can move your students toward reasonable, grade-level learning benchmarks in maximally effective ways. Designed to work with any curriculum, CBA will enable you to better understand and respond to your students’ learning needs and help you choose instructional activities that are best for your students.

There are six books in the CBA project:

- Cognition-Based Assessment and Teaching of Place Value
- Cognition-Based Assessment and Teaching of Addition and Subtraction
- Cognition-Based Assessment and Teaching of Multiplication and Division
- Cognition-Based Assessment and Teaching of Fractions
- Cognition-Based Assessment and Teaching of Geometric Shapes
- Cognition-Based Assessment and Teaching of Geometric Measurement
Any of these books can be used independently, though you may find it helpful to refer to several because the topics covered are interrelated.

**Critical Components of CBA**

The CBA approach emphasizes three key components that support students’ mathematical sense making and proficiency:

- clear, coherent, and organized research-based descriptions of students’ development of meaning for core ideas and reasoning processes in elementary school mathematics;
- assessment tasks that determine how each of student is reasoning about these ideas; and
- detailed descriptions of the kinds of instructional activities that will help students at each level of reasoning about these ideas.

More specifically, CBA includes the following essential components.

**Levels of Sophistication in Student Reasoning**

For many mathematical topics, researchers have found that students’ development of mathematical conceptualizations and reasoning can be characterized in terms of “levels of sophistication” (Battista, 2004; Battista and Clements, 1996; Battista et al., 1998; Cobb and Wheatley, 1988; Fuson et al., 1997; Steffe, 1988, 1992; van Hiele, 1986). Chapter 2 provides a framework that describes the development of students’ thinking and learning about fractions in terms of such levels. This framework describes the “cognitive terrain” in which students’ learning trajectories occur, including:

- the levels of sophistication that students pass through in moving from their intuitive ideas and reasoning to a more formal understanding of mathematical concepts;
- cognitive obstacles that students face in learning; and
- fundamental mental processes that underlie concept development and reasoning.

Figure 1 sketches the cognitive terrain that students must ascend to attain understanding of fractions. This terrain starts with students’ preinstructional reasoning about fractions, ends with a formal and deep understanding of fractions, and indicates the cognitive plateaus reached by students along the way. Not pictured in the sketch are sublevels of understanding that may exist at each plateau level. Note that students may travel slightly different trajectories in ascending through this cognitive terrain, and they may end their trajectories at different places, depending on the curricula and teaching they experience.
A Note About Student Work Samples

Chapter 2 includes many examples of students’ work, which are invaluable for understanding and using the levels. All of these examples are important because they show the rich diversity of student thinking at each level. However, the first time you work through the materials, you may want to read only a few examples for each type of reasoning—just enough examples to comprehend the basic idea of the level. Later, as you use the assessment tasks and instructional activities with your students, you can sharpen your understanding by examining additional examples both in the level descriptions and in the level examples for each assessment task.

Assessment Tasks

The Appendix contains a set of CBA assessment tasks that will enable you to determine your students’ mathematical thinking and precisely locate students’ positions in the cognitive terrain for learning fractions. These tasks not only assess exactly what students can do, but they reveal students’ reasoning and underlying mathematical cognitions. The tasks are followed by a description of what each level of reasoning might look like for each assessment task. These descriptions will help you pinpoint your students’ positions in the cognitive terrain of learning.

Using CBA assessment tasks to determine which levels of reasoning students are using will help you pinpoint students’ learning progress, know where students should proceed next in constructing meaning and competence for the idea, and decide which instructional activities will best promote students’ movement to higher levels of reasoning. It can also help guide your questions and responses in classroom discussions and in students’ small-group work. The CBA website at www.heinemann.com/products/E04345.aspx includes additional assessment tasks that you can use to further investigate your students’ understanding of fractions.
Instructional Suggestions

Chapter 3 provides suggestions for instructional activities that can help students progress to higher levels of reasoning. These activities are designed to meet the needs of students at each CBA level. The instructional suggestions are not meant to be comprehensive treatments of topics. Instead, they are intended to help you understand what kinds of tasks may help students make progress from one level or sublevel to the next higher level or sublevel.

Using the CBA Materials

Determining Students’ Levels of Sophistication

You can use CBA assessment tasks in several ways to determine students’ levels of sophistication in reasoning about fractions.

Individual Interviews

The most accurate way to determine students’ levels of sophistication is to administer the CBA assessment tasks in individual interviews with students. For many students, interviews make describing their thinking much easier—they are perfectly capable of describing their thinking orally but have difficulty doing it in writing. Individual interviews also enable teachers to ask probing questions at just the right time, which can be extremely helpful in revealing students’ thinking. (Beyond assessment purposes, the individual attention that students receive in individual assessment interviews provides students with added motivation, engagement, and learning.)

Whole-Class Discussion

In an “embedded assessment” model—in which assessment is embedded within instruction—you can give an assessment task to your whole class as an instructional activity. Each student should have a student sheet with the task on it. Students do all their work on their student sheets and describe in writing how they solve the task. When all the students have finished writing their descriptions of their solution methods, lead a class discussion of those methods. For instance, many teachers have a number of individual students present their solutions on an overhead projector or a document-projection device. As students describe their thinking, ask questions that encourage students to provide the detail you need to determine what levels of reasoning they are using. Also, at times, you can repeat or summarize students’ thinking in ways that model good explanations (but be sure to provide accurate

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1For helpful advice on scheduling and conducting student interviews, see Buschman (2001).
descriptions of what students say instead of formal versions of their reasoning). After each different student explanation, you can ask how many students used the strategy described. It is important that you not only have students orally describe their solution strategies but that you talk about how they can write and represent their strategies on paper. For instance, after a student has orally described his strategy, ask the class, “How could you describe this strategy on paper so that I would understand it without being able to talk to you?”

Another way to see if students’ written descriptions accurately describe their solution strategies is to ask students to come up to your desk and tell you individually what they did, which you can then compare to what they wrote.

Individual and Small-Group Work
You can also determine the nature of students’ reasoning by circulating around the room as students are working individually or in small groups on CBA assessment tasks or instructional activities. Observe student strategies and ask students to describe what they are doing as they are doing it. Seeing students actually work on problems often provides more accurate insights into what they are doing and thinking than merely hearing their explanations of their completed solutions (which sometimes do not match what they did). Also, as you talk to and observe students during individual or small-group problem solving, for students who are having difficulty accurately describing their work, write notes to yourself on students’ papers that tell you what they said and did (these notes are descriptive, not evaluative).

The Importance of Questioning
Keep in mind that the more students describe their thinking, the better they will become at describing that thinking, especially if you guide them toward providing increasingly accurate and detailed descriptions of their reasoning. For instance, consider a student working on the problem, “What is 1/2 + 1/3?” Suppose Jim writes “1/2 + 1/3 = 2/5” as his explanation of his strategy. Ask additional questions.

Teacher: *What did you do to figure out that 1/2 + 1/3 = 2/5?*

Jim: I added.

Teacher: *How did you add—show me.*

Jim: *I put 2 on the top and 5 on the bottom.*

Teacher: *How did you get your answer? Show me how you thought about this problem.*

Jim: I drew pictures.

Teacher: *Show me your pictures and what you did with them.*

Jim: I drew the fractions.
Teacher: Good. Explain how the picture showed you that the answer is 2/5.

Jim: There are 5 parts and 2 are shaded, that's 2/5.

Teacher: So on your paper, show your picture and write what you said about how the picture showed 2/5.

Listed below are some questions that can be helpful in conducting individual interviews, interacting with students during small-group work, or conducting a classroom discussion of an assessment task:

- That's interesting; tell me what you did.
- Tell me how you found your answer.
- How did you figure out this problem?
- I'd really like to understand how you're thinking; can you tell me more about it?
- Why did you do that?
- What were you thinking when you moved these objects?
- Did you check your answer to see whether it is correct? How?
- Explain your drawing to me.
- What do these marks that you made mean?
- What were you thinking when you did this part of the problem?
- What do you mean when you say . . . ?

Monitoring the Development of Students’ Reasoning

The CBA materials are designed to help you assess levels of reasoning, not levels of students. Indeed, a student might use different levels of reasoning on different tasks. For instance, a student might operate at a higher level when using physical materials such as place-value blocks than when she does not have physical materials to support her thinking. Also, a student might operate at different levels on tasks that are familiar to her, or that she has practiced, as opposed to tasks that are totally new to her. So, rather than attempting to assign a single level to a student, you should analyze a student's reasoning on several assessment tasks, then develop an overall
profile of how she is reasoning about the topic. An example of how this is done appears in Chapter 2.

To carefully monitor and even report to parents the development of student reasoning about particular mathematical topics, many teachers keep detailed records of students’ CBA reasoning levels during the school year. To do this, choose several CBA assessment tasks for each major mathematical topic you will cover during the year. Administer these tasks to all of your students either as individual interviews or as written work at several different times during the school year (for example, before and after each curriculum unit dealing with the topic). In addition to noting the tasks used and the date, record what levels each student used on the tasks.

Differentiating Instruction to Meet Individual Students’ Learning Needs

You can tailor instruction to meet individual students’ learning needs in several ways.

Individualized Instruction

The most effective way is to work with students individually, using the levels and tasks to precisely assess and guide students’ learning. This approach is an extremely powerful way to maximize an individual student’s learning.

Instruction by CBA Groups

Another effective way of meeting students’ needs is by putting students into groups based on their CBA levels of reasoning about a mathematical topic. You can then look to the instructional suggestions for tasks that will be maximally effective for helping the students in each group. For instance, you might have three or four groups in your class, each consisting of students who are reasoning at about the same CBA levels and need the same type of instruction.

Whole-Class Instruction

Another approach that many teachers have used successfully is selecting sets of tasks that all students in a class can benefit from doing. You do this by first determining the different levels of reasoning among students in the class. Then, as you consider possible instructional tasks, ask yourself these questions:

- “How will students at each level of reasoning attempt to do this task?”
- “Can students at different levels of reasoning succeed on the task by using different strategies?” (Avoid tasks that some students will not have any way of completing successfully.)
- “How will students at each level benefit by doing the task?”
“Will seeing how different students do the task help other students progress to higher levels of thinking because they are ready to hear new ways of reasoning about the task?”

Also, sets of tasks can be sequenced so that initial problems target students using lower levels of reasoning while later tasks target students using higher levels.

Another way to individualize whole-class instruction is to ask different questions to students at different levels as you circulate among students working in small groups. For instance, for students who use unequal pieces in determining fractions, you might ask, When you tried to find 1/3, did you divide the cake into 3 equal pieces? If we share the pieces for 3 people, will each person get the same amount of cake? Knowledge of CBA levels is invaluable in devising good questions and in asking appropriate questions for different students. In fact, when preparing to teach a lesson, many teachers use levels-of-sophistication descriptions to think about the kinds of questions they will ask students who are functioning at different levels.

Choosing which students to put into small groups for whole-class inquiry-based instruction is also important. If you think of your students’ CBA levels of reasoning on a particular type of task as being divided into three groups, you might put students in the high and middle groups together, or students in the middle and low groups together. Generally, putting students in the high and low groups together is not effective because their thinking is likely to be too different.

Assessment and Accountability

As a consequence of state and federal testing and accountability initiatives, most school districts and teachers are looking for materials and methods that will help them achieve state performance benchmarks. CBA is a powerful tool that can help you help your students achieve these benchmarks by:

- monitoring students’ development of reasoning about core mathematical ideas;
- identifying students who are having difficulties learning these ideas and diagnosing the nature of these difficulties;
- understanding the nature of weaknesses identified by annual state mathematics assessments results along with causes for these weaknesses; and
- understanding a framework for remediating student difficulties in conceptually and cognitively sound ways.

Moving Beyond Deficit Models

The CBA materials can help you move beyond the “deficit” model of traditional diagnosis and remediation. In the deficit model, teachers wait until students fail before attempting to diagnose and remediate their learning problems. CBA offers a more powerful, preventative model for helping students. By using CBA materials to
appropriately pretest students on core ideas that are needed for upcoming instructional units, you can identify which students need help and the nature of the help they need before they fail. By then using appropriate instructional activities, you can help students acquire the core knowledge needed to be successful in the upcoming units—making that instruction effective rather than ineffective for these students.

The Research Base

Not only have these materials gone through extensive field testing with both students and teachers, the CBA approach is consistent with major scientific theories describing how students learn mathematics with understanding. These theories agree that mathematical ideas must be personally constructed by students as they intentionally try to make sense of situations, and that to be effective, mathematics teaching must carefully guide and support students’ construction of personally meaningful mathematical ideas (Baroody and Ginsburg, 1990; Battista, 1999, 2001; Bransford, Brown, and Cocking, 1999; De Corte, Greer, and Verschaffel, 1996; Greeno, Collins, and Resnick, 1996; Hiebert and Carpenter, 1992; Lester, 1994; National Research Council, 1989; Prawat, 1999; Romberg, 1992; Schoenfeld, 1994; Steffe and Kieren, 1994; von Glasersfeld, 1995). Research shows that when students’ current ideas and beliefs are ignored, their development of mathematical understanding suffers. And conversely, “There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students’ changing conceptions as instruction proceeds” (Bransford et al., 1999, p. 11).

The CBA approach is also consistent with research on mathematics teaching. For instance, based on their research in the Cognitively Guided Instruction program, Carpenter and Fennema (1991) concluded that teachers must “have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes” (1991, p. 11). Indeed, a number of studies have shown that when teachers learn about such research on students’ mathematical thinking, they can use that knowledge in ways that positively influence their students’ mathematics learning (Carpenter et al., 1998; Cobb et al., 1991; Fennema and Franke, 1992; Fennema et al., 1996; Steff and D’Ambrosio, 1995). These materials will enable you to:

- develop a detailed understanding of your students’ current reasoning about specific mathematical topics, and
- choose learning goals and instructional activities to help your students build on their current ways of reasoning.

Indeed, these materials provide the kind of coherent, detailed, and well-organized research-based knowledge about students’ mathematical thinking that research has indicated is important for teaching (Fennema and Franke, 1992).
Research also shows that using formative assessment can produce significant learning gains in all students (Black and Wiliam, 1998). Furthermore, formative assessment can be especially helpful for struggling students, so it can reduce achievement gaps in mathematics learning. The CBA materials offer teachers a powerful type of formative assessment that monitors students’ learning in ways that enable teaching to be adapted to meet students’ learning needs. “For assessment to function formatively, the results have to be used to adjust teaching and learning” (Black and Wiliam, 1998, p. 142). To implement high-quality formative assessment, the major question that must be asked is, “Do I really know enough about the understanding of my pupils to be able to help each of them?” (Black and Wiliam, 1998, p. 143). CBA materials help answer this question.

**Using CBA Materials for RTI**

Response to Intervention (RTI) is a school-based, tiered prevention and intervention model for helping all students learn mathematics. Tier 1 focuses on high-quality classroom instruction for all students. Tier 2 focuses on supplemental, differentiated instruction to address particular needs of students within the classroom context. Tier 3 focuses on intensive individualized instruction for students who are not making adequate progress in Tiers 1 and 2.

CBA can be effectively used for all three RTI tiers. For Tier 1, CBA materials provide extensive, research-based descriptions of the development of students’ learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, CBA descriptions enable you to better understand and monitor each student’s mathematics learning through observation, embedded assessment, questioning, informal assessment during small-group work, and formal assessment. You can then choose instructional activities that meet your students’ learning needs—whole-class tasks that benefit students at all levels; different tasks for small groups of students at the same levels; and individualized supplementary student work. For Tier 3, CBA assessments and level-specific instructional suggestions provide road maps and directions for giving struggling students the long-term individualized instruction sequences they need.

**Supporting Students’ Development of Mathematical Reasoning**

CBA materials are designed to help students move to higher levels of reasoning. It is important, however, that instruction not demand that students “move up” the levels with insufficient cognitive support. Such demands result in students rote memorizing procedures that they cannot make personal sense of: jumps in levels are made internally by students, not by teachers or the curriculum. This does not
mean that students must progress through the levels without help. Teaching helps students by providing them with the right kinds of encouragement, support, and challenges—having students work on problems that stretch, but do not overwhelm, their reasoning, asking good questions, having them discuss their ideas with other students, and sometimes showing them ideas that they don’t invent themselves. But when we show students ideas, we should not demand that they use them. Instead, we should try to get students to adopt new ideas because they make personal sense of the ideas and see the new ideas as better than the ideas they currently possess.
In elementary school, a fraction is a symbolic expression of the form \( a/b \), where \( a \) and \( b \) are whole numbers and \( b \) is not zero (whole numbers are 0, 1, 2, 3 . . .). In junior high, fractions can be positive and negative, and later, in algebra, fractions can be more complicated expressions. Fractions describe quantities as portions or parts of wholes. They specify relationships between wholes and their parts. As numbers, fractions are numbers that are “in between” whole numbers.

It is difficult for students to move from working with whole numbers to working with fractions because a fractional quantity is described with two numbers, not one, and understanding a fraction requires one to explicitly comprehend a relationship between two quantities—the whole and its parts.

**Critical Components of Understanding Fractions**

There are several critical components of a genuine understanding of fractions. These components are discussed in the following sections.

**Partitioning**

Before students can understand fractions, they must understand partitioning. To partition a whole is to divide it into equal portions, like dividing a pizza equally among four people.

Being able to partition, however, does not mean that one understands fractions. For instance, a student might partition a pizza equally among four people but not understand how the pieces relate to the whole, saying simply that each person gets one piece.
Fractions

To understand fractions, students must be able to partition a whole into equal portions and understand how the portions are related to the whole. Students must also understand how fractional quantities are symbolized mathematically. So, in the fraction \( \frac{a}{b} \), \( b \) (the denominator) denotes how many equal parts are in the whole, and \( a \) (the numerator) denotes how many equal parts are in the fractional quantity specified by \( \frac{a}{b} \).

Iteration

Deeper understanding of the role of partitioning in fractions comes from understanding the complementary process of iteration. Partitioning starts with the whole and divides it into equal parts. Iteration starts with a part and repeats it to make the whole. Students take a major step toward substantive understanding of fractions when they understand the relationship between the processes of iteration and partitioning: iterating a piece to make a whole defines a partitioning of the whole; partitioning a whole into equal pieces defines an iteration that makes the whole.

To use iteration to create a fraction \( \frac{n}{d} \) of a shape, you find a part that when iterated \( d \) times makes the whole, and then iterate that part \( n \) times.

Understanding Unit Fractions

To see how iteration is used to understand unit fractions, consider the question: What fraction of the whole shape below is shaded?

Because the shaded rhombus can be iterated 4 times to make the whole shape, the rhombus is \( \frac{1}{4} \) of the whole shape.
Iteration partitions the whole shape into 4 equal pieces and establishes a relationship between the shaded part and the whole—it takes 4 rhombuses to make the whole shape. Importantly, students must do more than simply iterate the rhombus; they must explicitly recognize that this iteration partitions the whole into 4 equal parts.

**Understanding Non-Unit Fractions**
Once students understand unit fractions, we would like them to understand non-unit, proper fractions. Again, iteration is critical. For instance, suppose we want to determine what fractional part of the whole shape below is shaded.

Because iterating the rhombus 4 times makes the whole shape, and iterating it 3 times makes the shaded part, the shaded part is $\frac{3}{4}$ of the whole.

Beyond the physical iteration process, we would also like students to understand that non-unit fractions are made up of iterations of unit fractions. For instance, $\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ or $\frac{3}{8} = 3 \times \frac{1}{8}$.

**Understanding Improper Fractions and Mixed Numerals**
Students can move from using iteration to understand proper fractions to using it to understand improper fractions and mixed numerals. A *proper fraction* is a fraction
in which the numerator is less than or equal to the denominator (e.g., 5/6, 3/3). An improper fraction is a fraction in which the numerator is greater than the denominator (e.g., 6/5). A mixed number has both a whole number and a fractional part (e.g., 3 4/5).

To see how iteration can be used to establish meaning for an improper fraction, consider the diagram below. Because we know that 4 iterations of the shaded rhombus makes the whole white shape, 9 iterations of the rhombus represents the improper fraction 9/4 or 2 1/4 of this whole.

![Diagram](image)

Because proper fractions deal with just one whole, and improper fractions and mixed numbers deal with more than one whole, it is easier for students to understand proper fractions than it is for them to understand improper fractions and mixed numbers.

**Equivalent Fractions, Comparisons, and Operations**

Another essential idea in understanding fractions is equivalent fractions. Two fractions are equivalent if they specify the same quantity. For example, 3/4 and 6/8 are equivalent because they specify the same amount of shading of the whole.

![Diagram](image)

The difference in these fractions is that 3/4 specifies the quantity in terms of fourths (the rhombuses), and 6/8 specifies this same quantity in terms of eighths (the triangles). The fact that the same quantity can be named in different ways initially can be puzzling to students.
Another critical idea in dealing with fractions is that of creating fractions with common denominators. Two fractions have a common denominator if their denominators are equal (e.g., $\frac{2}{7}$ and $\frac{4}{7}$). When two fractions have a common denominator, their iterated parts are the same size, which makes it easier to compare, add, and subtract them. Consequently, to compare, add, and subtract unlike fractions (those with different denominators), we convert them to fractions with common denominators. For example, to find which is larger, $\frac{2}{3}$ or $\frac{5}{7}$, we convert both fractions to their equivalents using 21 as the common denominator.

$$\frac{2}{3} = \frac{14}{21} \quad \frac{5}{7} = \frac{15}{21}$$

So, because $\frac{14}{21} < \frac{15}{21}$, we know that $\frac{2}{3} < \frac{5}{7}$.

Similarly, to add $\frac{1}{4}$ and $\frac{1}{8}$, we convert fourths to eighths, which gives $\frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$.

**Fractions of Sets**

In general, students understand fractions of whole geometric regions before they understand fractions of sets of objects or fractions of numbers. To see why, consider the problem of finding $\frac{5}{9}$ of 36 dots.
To determine one-ninth of 36, students must find a set of dots that when iterated 9 times makes 36 (i.e., they must partition 36 into 9 equivalent sets). Using objects, drawings, skip-counting, or arithmetic (36 ÷ 9 = 4), students must determine that they can iterate 4 dots 9 times to make 36. This iteration partitions 36 into 9 sets of 4 and shows that 1/9 of the set of 36 dots is 4 dots.

To find 5/9 of 36 dots, students must iterate a set of 4 dots 5 times. This second iteration shows that 5/9 of 36 dots is 20 dots.

As demonstrated above, finding a fraction of a set of objects is complex because students must define and maintain a set of objects as the whole (in this case 36 dots) and then must iterate a composite unit of objects (a set of 4 dots in this case). This is considerably more difficult than, say, finding 3/4 of a square, in which the whole is one shape (the given square) and a single shape (a smaller square) must be iterated.
Fractions on Number Lines

Mathematicians frequently use number lines to represent and reason about numbers. Consequently, students in mathematics in and beyond algebra must often use number lines, so it is important for elementary school students to start learning about this representational tool. However, understanding number lines does not come easily for many students. To see why, let us review the steps used to represent numbers on the number line, keeping in mind that this representation is based on iterating and measuring lengths:

**Step 1.** Pick an arbitrary point on the line and label it zero.

**Step 2.** Pick another arbitrary point on the line and label it 1.

**Step 3.** Iterate the unit length between 0 and 1 in both directions to find the positions for all the other integers (±1, ±2, ±3,…).

**Step 4.** Determine the positions of fractions (rational numbers) by partitioning unit lengths into equal pieces and that can be iterated to form the whole.

Note that the choice of the point for 1 determines the positions of all other numbers, the positive direction on the number line, and the unit length (the distance between all pairs of consecutive integers is equal to the distance between 0 and 1).

A special difficulty that many students have with representing fractions on the number line arises from their lack of understanding that numbers are represented on the number line by points, not segments. For instance, 1/3 is represented by the point at the right end of a segment that starts at 0 and has length equal to 1/3 of the unit length.

Two-thirds is represented by the right endpoint of a segment that results from iterating a 1/3 segment twice.

Understanding fractions as points instead of segments on a line is a big change for many students. For example, consider a student who, when asked to show 1/3 on the number line, wrote the following.

This student correctly uses her knowledge of partitioning objects to see how to divide the segment from 0 to 1 into thirds. But she sees 1/3 as a segment, not as the endpoint of a line segment starting at 0 and ending at 1/3.
Another difficulty students have with number lines arises from their lack of understanding of linear measurement. Such students count hash marks on the number line as objects in and of themselves rather than as indicators of iterations of the unit length. For example, when asked what number is represented by X on the number line below, one student counted visible hash marks and said, “There’s 7 of these [hash marks]; X is at 3. So X is 3/7.”

Making the Unit Explicit

Fractional amounts are always relative to a unit (or whole, or 1). Many students struggle to develop a strong conceptualization of fractions because they have difficulty establishing and maintaining the unit. To help students better understand the role of the unit in fractions, it is important to make this role explicit in classroom discussions. For instance, 1/2 of a pizza or rectangle is greater than 1/3 of the same pizza or rectangle.

Similarly, 1/2 of a set of 12 dots is greater than 1/3 of the same set of 12 dots. More abstractly, if we say that 1/2 is larger than 1/3, we really mean that 1/2 of the unit or 1 is greater than 1/3 of the same unit or 1.

However, if the unit changes, then statements like “1/2 is greater than 1/3” can get complicated. For instance, as shown below, 1/2 of Rectangle 2 is smaller than 1/3 of Rectangle 3.
There is a subtle, but extremely important issue here that may underlie some fundamental student misconceptions. In many discussions by accomplished users of fractions (including many textbook explanations), when they say “one-half” without any further specification (such as one-half of something specific), the unit or 1 is implicit or “assumed understood” rather than explicit. When we are talking about pure numbers, 1/2 always equals 1/2.

However, when we are talking about 1/2 of something—like pizzas, or sets of cupcakes, or even numbers (1/2 of 10 versus 1/2 of 30)—1/2 may be unequal to 1/2 because it is 1/2 of one thing versus 1/2 of a different thing. So be careful with language. Whenever you refer to 1/2 “of something,” be sure that you include the something in the reference—say, “one-half of this pizza,” not “one-half.”

Similarly, be explicit about what the unit is when adding or subtracting fractions. For example (continuing with the above pictures), 1/2 of Rectangle 1 + 1/3 of Rectangle 1 makes 5/6 of Rectangle 1. We keep the rectangle constant throughout the addition.

Understanding Students’ Levels of Sophistication for Fractions

The CBA levels provide a detailed description of the development of students’ reasoning about fractions. This detail is critical for tailoring instruction to meet students’ learning needs. However, when you are first learning a set of CBA levels, the amount of detail can be overwhelming. So, keep in mind that understanding CBA levels comes in stages and develops over time. First, you will learn the major features of the levels-of-sophistication framework for fractions. As you use CBA with your students, you will learn the details of the framework.

Zooming Out to Get an Overview

To begin understanding the CBA levels for fractions, it is important to develop an understanding of the overall organization of the levels. The chart on page 10 shows the CBA fractions levels in a “zoomed out” view.
These broad levels describe the major ways students reason about fractions. The
levels suggest an overall learning sequence: in Levels 1–3 students develop limited
conceptualizations of fractions, which in Level 4 matures into a more general and
abstract conceptualization. In Levels 5–7 students develop increasingly sophisticated
reasoning about arithmetic operations on fractions.

Understanding Algorithms

A computational algorithm is a precisely specified sequence of actions performed on
written symbols that systematically solves one general type of computational problem.

The levels of sophistication in Cognition-Based Assessment (CBA) describe
students’ development of core concepts and ways of reasoning about fractions. An
important part of this development is understanding and becoming fluent with using
computational algorithms. However, if algorithms are taught too early in students’ devel-
opment of reasoning about fractions, students cannot understand the algorithms conceptu-
ally, so they learn them by rote. Indeed, most students in traditional instruction learn
traditional algorithms for fractions rote, without understanding the underlying
number properties. Students who learn computational algorithms for fractions before
achieving Level 5 reasoning will learn them rote.

Zooming in to Meet Individual Students’ Needs

Understanding individual students’ reasoning precisely enough to maximize their
learning or remediate a learning difficulty requires a detailed picture. We must
“zoom in” to see sublevels (see Figure 1.1). The “jumps” between sublevels must be
small enough that students can achieve them with small amounts of instruction in relatively short periods of time.

Imagine students trying to climb to a cognitive plateau needed to meet an instructional goal. In Situation A, the student has to make a cognitive jump that is too great. In Situation B, the student can get to the goal by using accessible CBA sublevels as stepping-stones. To provide students the instructional guidance and cognitive support they need to develop a thorough understanding of mathematical ideas, you need to understand and use the sublevels. Chapter 2 provides detailed descriptions and illustrations of all the CBA levels and sublevels for fractions.

**CBA Levels and Preparation for Algebra**

CBA levels focus on the development of concepts and reasoning, in addition to computational fluency. However, students often learn computational algorithms for fractions by rote, without understanding the underlying number properties. This rote learning not only hinders students’ development of understanding of computation, it deprives them of foundational ideas needed to understand and master algebra. The CBA levels of sophistication focus on students’ development of this foundational numerical reasoning. These levels trace students’ development from intuitive concepts and reasoning to the number-property-based fluency with computational procedures that serves as the conceptual foundation for algebraic reasoning.