COGNITION-BASED ASSESSMENT & TEACHING

of Multiplication and Division
Michael T. Battista

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Building on Students’ Reasoning

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Portsmouth, NH
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—Michael Battista
Introduction

Traditional mathematics instruction requires all students to learn a fixed curriculum at the same pace and in the same way. At any point in traditional curricula, instruction assumes that students have already mastered earlier content and, based on that assumption, specifies what and how students should learn next. The sequence of lessons is fixed; there is little flexibility to meet individual student’s learning needs. Although this approach appears to work for the top 20 percent of students, it does not work for the other 80 percent (Battista, 1999, 2001). And even for the top 20 percent of students, the traditional approach is not maximally effective (Battista, 1999, 2001). For many students, traditional instruction is so distant from their needs that each day they make little or no learning progress and fall farther and farther behind curriculum demands. In contrast, Cognition-Based Assessment (CBA) offers a cognition-based framework to support teaching that enables all students to understand, make personal sense of, and become proficient with mathematics.

The CBA approach to teaching mathematics focuses on deep understanding and reasoning, within the context of continually assessing and understanding students’ mathematical thinking then builds on that thinking instructionally. Rather than teaching predetermined, fixed content at times when it is inaccessible to many students, the CBA approach focuses on maximizing individual student progress no matter where students are in their personal development. As a result, you can move your students toward reasonable, grade-level learning benchmarks in maximally effective ways. Designed to work with any curriculum, CBA will enable you to better understand and respond to your students’ learning needs and help you choose instructional activities that are best for your students.

There are six books in the CBA project:

- Cognition-Based Assessment and Teaching of Place Value
- Cognition-Based Assessment and Teaching of Addition and Subtraction
- Cognition-Based Assessment and Teaching of Multiplication and Division
- Cognition-Based Assessment and Teaching of Fractions
- Cognition-Based Assessment and Teaching of Geometric Shapes
- Cognition-Based Assessment and Teaching of Geometric Measurement
Any of these books can be used independently, though you may find it helpful to refer to several because the topics covered are interrelated.

Critical Components of CBA

The CBA approach emphasizes three key components that support students’ mathematical sense making and proficiency:

- clear, coherent, and organized research-based descriptions of students’ development of meaning for core ideas and reasoning processes in elementary school mathematics;
- assessment tasks that determine how each student is reasoning about these ideas; and
- detailed descriptions of the kinds of instructional activities that will help students at each level of reasoning about these ideas.

More specifically, CBA includes the following essential components.

Levels of Sophistication in Student Reasoning

For many mathematical topics, researchers have found that students’ development of mathematical conceptualizations and reasoning can be characterized in terms of “levels of sophistication” (Battista, 2004; Battista and Clements, 1996; Battista et al., 1998; Cobb and Wheatley, 1988; Fuson et al., 1997; Steffe, 1988, 1992; van Hiele, 1986). Chapter 2 presents a framework that describes the development of students’ thinking and learning about multiplication and division in terms of these levels. This framework describes the “cognitive terrain” in which students’ learning trajectories occur, including:

- the levels of sophistication that students pass through in moving from their intuitive ideas and reasoning to a more formal understanding of mathematical concepts;
- cognitive obstacles that students face in learning; and
- fundamental mental processes that underlie concept development and reasoning.

Figure 1 sketches the cognitive terrain that students must ascend to attain understanding of multiplication and division of whole numbers. This terrain starts with students’ preinstructional reasoning about multiplication and division, ends with a formal and deep understanding of multiplication and division, and indicates the cognitive plateaus reached by students along the way. Not pictured in the sketch are sublevels of understanding that may exist at each plateau level. Note that students may travel slightly different trajectories in ascending through this cognitive terrain, and they may end their trajectories at different places depending on the curricula and teaching they experience.
A Note About the Student Work Samples

Chapter 2 includes many examples of students’ work, which are invaluable for understanding and using the levels. All of these examples are important, for they show the rich diversity of student thinking at each level. However, the first time you work through the materials, you may want to read only a few examples for each type of reasoning—just enough examples to comprehend the basic idea of the level. Later, as you use the assessment tasks and instructional activities with your students, you can sharpen your understanding by examining additional examples both in the level descriptions and in the level examples for each assessment task.

Assessment Tasks

The Appendix contains a set of CBA assessment tasks that will enable you to determine your students’ mathematical thinking and precisely locate students’ positions in the cognitive terrain for learning that idea. These tasks not only assess exactly what students can do, they also reveal students’ reasoning and underlying mathematical cognitions. The tasks are followed by a description of what each level of reasoning might look like for each assessment task. These descriptions will help you pinpoint your students’ positions in the cognitive terrain of learning.

Using CBA assessment tasks to determine which levels of sophistication students are using will help you pinpoint students’ learning progress, know where students should proceed next in constructing meaning and competence for the idea, and decide which instructional activities will best promote students’ movement to higher levels of reasoning. It can also help guide your questions and responses in classroom discussions and in students’ small-group work. The CBA website at www.heinemann.com/products/E01271.aspx includes additional assessment tasks that you can use to further investigate your students’ understanding of multiplication and division.
Instructional Suggestions

Chapters 3 and 4 provide suggestions for instructional activities that can help students progress to higher levels of reasoning. These activities are designed to meet the needs of students at each CBA level. The instructional suggestions are not meant to be comprehensive treatments of topics. Instead, they are intended to help you understand what kinds of tasks may help students make progress from one level/sublevel to the next higher level/sublevel.

Using the CBA Materials

Determining Students’ Levels of Sophistication

There are several ways that you can use CBA assessment tasks to determine students’ levels of sophistication in reasoning about multiplication and division.

Individual Interviews

The most accurate way to determine students’ levels of sophistication is to administer the CBA assessment tasks in individual interviews with students. For many students, interviews make describing their thinking much easier; they are perfectly capable of describing their thinking orally but have difficulty doing it in writing. Individual interviews also allow teachers to ask probing questions at just the right time, which can be extremely helpful in revealing students’ thinking. (Beyond assessment purposes, the individual attention that students receive in individual assessment interviews can also provide students with added motivation, engagement, and learning.)

Whole-Class Discussion

In an “embedded assessment” model—in which assessment is embedded within instruction—you can give an assessment task to your whole class as an instructional activity. Each student should have a student sheet with the task on it. Students do all their work on their student sheets and describe in writing how they solve the task. When all the students are finished writing their descriptions of their solution methods, have a class discussion of those methods. For instance, many teachers have a number of individual students present their solutions on an overhead projector or a document-projection device. As students describe their thinking, ask questions that encourage students to provide the detail you need to determine what levels of reasoning they are using. Also, at times, you can revoice or summarize students’ thinking in ways that model good explanations (but be sure that you provide accurate descriptions of what students say instead of formal versions of their reasoning). After

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each different student explanation, you can ask how many students used the strategy described. It is important that you not only have students orally describe their solution strategies but that you talk about how they can write and represent their strategies on paper. For instance, after a student has orally described his strategy, ask the class, “How could you describe this strategy on paper so that I would understand it without being able to talk to you?”

Another way to see if students’ written explanations accurately describe their solution strategies is to ask students to come up to your desk and tell you individually what they did, which you can then compare to what they wrote.

**Individual and Small-Group Work**

You can also determine the nature of students’ reasoning by circulating around the room as students are working individually or in small groups on CBA assessment tasks or instructional activities. Observe student strategies and ask students to describe what they are doing as they are doing it. Seeing students actually work on problems often provides more accurate insights into what they are doing and thinking than merely hearing their explanations of their completed solutions (which sometimes do not match what they did). Also, as you talk to and observe students during individual or small-group problem solving, for students who are having difficulty accurately describing their work, write notes to yourself on students’ papers that tell you what they said and did (these notes are descriptive, not evaluative).

**The Importance of Questioning**

Keep in mind that the more students describe their thinking, the better they will become at explaining that thinking, especially if you guide them toward providing increasingly accurate and detailed descriptions of their reasoning. For instance, if a student says, “I counted,” ask, “How did you count? Count out loud to show me what you did. How could you write about what you did?”

As a more specific example, consider a student working on the problem, “Mary has 4 bags and 5 apples in each bag. How many apples does Mary have?” Suppose Jim writes “4 × 5 = 20” as his explanation of his strategy. Ask additional questions.

**Teacher:** What did you do to figure out that 5 × 4 = 20?

**Jim:** I counted.

**Teacher:** How did you count? Count out loud for me.

**Jim:** 5, 10, 15, 20.

**Teacher:** Okay, that’s a great way to solve the problem. How could you write that on your sheet?

**Jim:** I could write that I counted.

**Teacher:** Great. And what else could you write so I know how you counted?
Jim: I don't know.

Teacher: What numbers did you say when you counted?

Jim: 5, 10, 15, 20.

Teacher: So, you could write these numbers on your sheet.

Listed below are some questions that can be helpful in conducting individual interviews, interacting with students during small-group work, or conducting a classroom discussion of an assessment task.

- That's interesting; tell me what you did.
- Tell me how you found your answer.
- How did you figure out this problem?
- I'd really like to understand how you're thinking; can you tell me more about it?
- Why did you do that?
- What were you thinking when you moved these objects?
- Did you check your answer to see whether it is correct? How?
- Explain your drawing to me.
- What do these marks that you made mean?
- What were you thinking when you did this part of the problem?
- What do you mean when you say…?

Monitoring the Development of Students' Reasoning

The CBA materials are designed to help you assess levels of reasoning, not levels of students. Indeed, a student might use different levels of reasoning on different tasks. For instance, a student might operate at a higher level when using physical materials such as place-value blocks than when she does not have physical materials to support her thinking. Also, a student might operate at different levels on tasks that are familiar to her or that she has practiced as opposed to tasks that are totally new to her. So, rather than attempting to assign a single level to a student, you should analyze a student's reasoning on several assessment tasks then develop an overall profile of how she is reasoning about the topic. An example of how this is done appears in Chapter 2.

To carefully monitor and even report to parents the development of student reasoning about particular mathematical topics, many teachers keep detailed records of students’ CBA reasoning levels during the school year. To do this, choose several CBA assessment tasks for each major mathematical topic you will cover during the year. Administer these tasks to all of your students either as individual interviews or as written work at several different times during the school year (say, before and after each curriculum unit dealing with the topic). In addition to noting the tasks used and the date, record what levels each student used on the tasks.
Differentiating Instruction to Meet Individual Students’ Learning Needs

You can tailor instruction to meet individual students’ learning needs in several ways.

**Individualized Instruction**

The most effective way to meet students’ learning needs is to work with them individually using the levels and tasks to precisely assess and guide students’ learning. This approach is an extremely powerful way to maximize an individual student’s learning.

**Instruction by CBA Groups**

Another effective way of meeting students’ needs is putting students into groups based on their CBA levels of reasoning about a mathematical topic. You can then look to the instructional suggestions for tasks that will be maximally effective for helping the students in each group. For instance, you might have three or four groups in your class, each consisting of students who are reasoning at about the same CBA levels and need the same type of instruction.

**Whole-Class Instruction**

Another approach that many teachers have used successfully is selecting sets of tasks that all students in a class can benefit from doing. You do this by first determining the different levels of reasoning among students in the class. Then, as you consider possible instructional tasks, ask yourself,

- “How will students at each level of reasoning attempt to do this task?”
- “Can students at different levels of reasoning succeed on the task by using different strategies?” (Avoid tasks that some students will not have any way of completing successfully.)
- “How will students at each level benefit by doing the task?”
- “Will seeing how different students do the task help other students progress to higher levels of thinking because they are ready to hear new ways of reasoning about the task?”

Also, sets of tasks can be sequenced so that initial problems target students using lower levels of reasoning while later tasks target students using higher levels.

Another way to individualize whole-class instruction is to ask different questions to students at different levels as you circulate among students working in small groups. For instance, for students who are operating on numbers as collections of ones, you might ask if there is another way to count to solve the problem—can they use skip-counting? On the same problem, for students who are already skip-counting, you might ask if they can do the problem without counting (say, by using...
number properties and derived facts). Knowledge of CBA levels is invaluable in devising good questions and in asking appropriate questions for different students. In fact, when preparing to teach a lesson, many teachers use levels-of-sophistication descriptions to think about the kinds of questions they will ask students who are functioning at different levels.

Choosing which students to put into small groups for whole-class inquiry-based instruction is also important. If you think of your students’ CBA levels of reasoning on a particular type of task as being divided into three groups, you might put students in the high and middle groups together or students in the middle and low groups together. Generally, putting students in the high and low groups together is not effective because their thinking is likely to be too different.

Assessment and Accountability

As a consequence of state and federal testing and accountability initiatives, most school districts and teachers are looking for materials and methods that will help them achieve state performance benchmarks. CBA is a powerful tool that can help you help your students achieve these benchmarks by:

- monitoring students’ development of reasoning about core mathematical ideas;
- identifying students who are having difficulties learning these ideas and diagnosing the nature of these difficulties;
- understanding the nature of weaknesses identified by annual state mathematics assessment results along with causes for these weaknesses; and
- understanding a framework for remediating student difficulties in conceptually and cognitively sound ways.

Moving Beyond Deficit Models

The CBA materials can help you move beyond the “deficit” model of traditional diagnosis and remediation. In the deficit model, teachers wait until students fail before attempting to diagnose and remediate their learning problems. CBA offers a more powerful, preventive model for helping students. By using CBA materials to appropriately pretest students on core ideas that are needed for upcoming instructional units, you can identify which students need help and the nature of the help they need before they fail. By then using appropriate instructional activities, you can help students acquire the core knowledge needed to be successful in the upcoming units—making that instruction effective rather than ineffective for these students.

The Research Base

Not only have these materials gone through extensive field testing with both students and teachers, but the CBA approach is also consistent with major scientific theories
describing how students learn mathematics with understanding. These theories agree that mathematical ideas must be personally constructed by students as they intentionally try to make sense of situations. Furthermore, to be effective, mathematics teaching must carefully guide and support students’ construction of personally meaningful mathematical ideas (Baroody and Ginsburg, 1990; Battista, 1999, 2001; Bransford, Brown, and Cocking, 1999; De Corte, Greer, and Verschaffel, 1996; Greeno, Collins, and Resnick, 1996; Hiebert and Carpenter, 1992; Lester, 1994; National Research Council, 1989; Prawat, 1999; Romberg, 1992; Schoenfeld, 1994; Steffe and Kieren, 1994; von Glasersfeld, 1995). Research shows that when students’ current ideas and beliefs are ignored, their development of mathematical understanding suffers. And conversely, “There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students’ changing conceptions as instruction proceeds” (Bransford et al., 1999, p. 11).

The CBA approach is also consistent with research on mathematics teaching. For instance, based on their research in the Cognitively Guided Instruction program, Carpenter and Fennema concluded that teachers must “have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes” (1991, p. 11). Indeed, a number of studies have shown that when teachers learn about such research on students’ mathematical thinking, they can use that knowledge in ways that positively impact their students’ mathematics learning (Carpenter et al., 1998; Cobb et al., 1991; Fennema and Franke, 1992; Fennema et al., 1996; Steff and D’Ambrosio, 1995). These materials will enable you to:

- develop a detailed understanding of your students’ current reasoning about specific mathematical topics and
- choose learning goals and instructional activities to help your students build on their current ways of reasoning.

Indeed, these materials provide the kind of coherent, detailed, and well-organized research-based knowledge about students’ mathematical thinking that research has indicated is important for teaching (Fennema and Franke, 1992).

Research also shows that using formative assessment can produce significant learning gains in all students (Black and Wiliam, 1998). Furthermore, formative assessment can be especially helpful for struggling students, so it can reduce achievement gaps in mathematics learning. The CBA materials offer teachers a powerful type of formative assessment that monitors students’ learning in ways that enable teaching to be adapted to meet students’ learning needs. “For assessment to function formatively, the results have to be used to adjust teaching and learning” (Black and Wiliam, 1998, p. 142). To implement high-quality formative assessment, the major question that must be asked is, “Do I really know enough about the understanding of my pupils to be able to help each of them?” (Black and Wiliam, 1998, p. 143). CBA materials help answer this question.
Using CBA Materials for RTI

Response to Intervention (RTI) is a school-based, tiered prevention and intervention model for helping all students learn mathematics. Tier 1 focuses on high-quality classroom instruction for all students. Tier 2 focuses on supplemental, differentiated instruction to address particular needs of students within the classroom context. Tier 3 focuses on intensive individualized instruction for students who are not making adequate progress in Tiers 1 and 2.

CBA can be effectively used for all three RTI tiers. For Tier 1, CBA materials provide extensive, research-based descriptions of the development of students’ learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, CBA descriptions enable you to better understand and monitor each student’s mathematics learning through observation, embedded assessment, questioning, informal assessment during small-group work, and formal assessment. You can then choose instructional activities that meet your students’ learning needs—whole-class tasks that benefit students at all levels; different tasks for small groups of students at the same levels; individualized supplementary student work. For Tier 3, CBA assessments and level-specific instructional suggestions provide road maps and directions for giving struggling students the long-term individualized instruction sequences they need.

Supporting Students’ Development of Mathematical Reasoning

CBA materials are designed to help students move to higher levels of reasoning. It is important, however, that instruction not demand that students “move up” the levels with insufficient cognitive support. Such demands result in students rote memorizing procedures that they cannot make personal sense of. Jumps in levels are made internally by students, not by teachers or the curriculum. This does not mean that students must progress through the levels with no help. Teaching helps students by providing them with the right kinds of encouragement, support, and challenges—having students work on problems that stretch, but do not overwhelm, their reasoning, asking good questions, having them discuss their ideas with other students, and sometimes showing them ideas that they don’t invent themselves. But when we show students ideas, we should not demand that they use them. Instead, we should try to get students to adopt new ideas because students make personal sense of the ideas and see the new ideas as better than the ideas they currently have.
There are two critical components in the development of students’ reasoning about multiplication and division of whole numbers. First, students must understand what the operations mean and recognize when each is appropriate in problem solving. Second, students must understand and become proficient with strategies for performing computations for these operations.

What It Means to Multiply and Divide

In typical multiplication problems, we are given the number of equivalent groups and the number in each group and asked to find the total. In typical division problems, we are given the total, or product, and one of the factors (number of equivalent groups or the number in each group) and asked to find the other factor.

There are two major division situations. In measurement division, we are given the total number of objects and asked how many equivalent groups of a given size can be made from the total (for example, how many groups of 3 are in 15, or what is the measure of 15 if we take 3 as the unit of measure?). In partitive division, we are given the total number of objects and asked how many objects will be in each group if the objects are partitioned (divided, separated) into a given number of equivalent groups (for example, if 15 tennis balls are partitioned into 5 equal groups, how many will be in each group?).

The major meanings for multiplication and division are summarized in Figure 1.1. Note that although the examples describe situations that focus on sets of discrete objects (e.g., 5 tennis balls), the same situations can occur for continuous quantities (e.g., 5 inches), which can be more difficult for students to conceptualize.
Many students are puzzled by the two different meanings for division. It is often helpful to these students to see how the measurement and partitive meanings are related for a specific problem. To see how the measurement meaning applies to the partitive division example in Figure 1.1, think of putting 1 ball into each of 5 cans 3 times (see Figure 1.2). The first time, we distribute 5 balls, 1 into each of 5 cans; we subtract 5 from 15 to see that there are 10 remaining balls. The second time, we distribute 5 balls, 1 into each of 5 cans, and subtract 5 from 10 balls to see that there are 5 remaining balls. The third time we distribute 5 balls, 1 into each of 5 cans, and then subtract 5 balls to see that we have none left. When we ask how many balls are in each can (partitive), we are also asking how many sets of 5 are in 15 or how many times we can subtract 5 from 15 (measurement). So for a set of 15 objects, the number of objects in a group if there are 5 equal groups equals the number of groups of 5 objects.

**Relationship Between Measurement and Partitive Division**

Many students are puzzled by the two different meanings for division. It is often helpful to these students to see how the measurement and partitive meanings are related for a specific problem. To see how the measurement meaning applies to the partitive division example in Figure 1.1, think of putting 1 ball into each of 5 cans 3 times (see Figure 1.2). The first time, we distribute 5 balls, 1 into each of 5 cans; we subtract 5 from 15 to see that there are 10 remaining balls. The second time, we distribute 5 balls, 1 into each of 5 cans, and subtract 5 from 10 balls to see that there are 5 remaining balls. The third time we distribute 5 balls, 1 into each of 5 cans, and then subtract 5 balls to see that we have none left. When we ask how many balls are in each can (partitive), we are also asking how many sets of 5 are in 15 or how many times we can subtract 5 from 15 (measurement). So for a set of 15 objects, the number of objects in a group if there are 5 equal groups equals the number of groups of 5 objects.
**Multiplication as Iterating Numbers**

Students begin to develop an understanding of multiplication by iterating (repeating and accumulating) numerical composite units. (A **composite unit** is a collection of things that has been mentally combined and treated as a unit.) For instance, to find the total in 5 groups of 3, we can create five instances of the composite 3 then count all the objects by ones (see Figure 1.3).

A more sophisticated way of solving this problem is to enumerate subtotals after each iteration by skip-counting (3, 6, 9, 12, 15). Iteration 1 gives the total in 1 composite of 3, iteration 2 gives the total in 2 composites of 3, iteration 3 gives the total in 3 composites of 3, and so on. The successive totals in enumerating composites of 3 are the skip-counts of 3.

![Figure 1.3](ch01_MD_6078.indd.png)

**Multiplication as Coordinated Counting**

To use iterative reasoning for multiplication, students must be able to coordinate simultaneous counting sequences—one for the number of objects in a group, one for the number of groups, and one for the total number of objects in accumulating iterations of groups. For example, to enumerate 5 groups of 3, a student might think, 1, 2, 3 (that’s 1 group); 4, 5, 6 (that’s 2 groups); 7, 8, 9 (that’s 3 groups); 10, 11, 12 (that’s 4 groups), 13, 14, 15 (that’s 5 groups). The student is coordinating the count for the total number of objects (1–15), the count for the number of groups (1 group, 2 groups, ... 5 groups), and the count of 3 in each group (1, 2, 3 for objects in the first group; 4, 5, 6 for objects in the second group; and so on).

Coordinating these counting sequences is difficult, and lack of coordination is a major source of student errors. Indeed, as the factors get larger, it becomes more and more difficult for students to keep track of these counting sequences. So, for larger numbers, students need to move on to more sophisticated strategies for thinking about iteration and multiplication.
Relationship Between Multiplication and Division

In addition to understanding the various meanings of multiplication and division, students must also understand the critically important inverse relationship between multiplication and division. One way of expressing this inverse relationship is:

\[ a \div b = c \text{ if and only if } a = b \times c \]

<table>
<thead>
<tr>
<th>Multiplication:</th>
<th>factor 1 ( \times ) factor 2 = product</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 ( \times ) 5 = 15</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Division:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(multiplication language)</td>
</tr>
<tr>
<td>dividend ( \div ) divisor = quotient</td>
</tr>
<tr>
<td>15 ( \div ) 5 = 3</td>
</tr>
</tbody>
</table>

Another way of expressing that multiplication and division are inverse operations is:

\[ (n \times a) \div a = n \text{ or } (n \div a) \times a = n \]

This expression shows that multiplication and division “undo” each other. If we start with 3 and multiply by 5, we get 15. To undo the effect on 3 of multiplying by 5, divide 15 by 5 and we get back to 3.

A very useful consequence of the inverse relationship between multiplication and division is that we can solve division problems by solving related multiplication problems. For instance, to solve 45 ÷ 15, we can ask: What number times 15 gives 45? For this reason, and because multiplicative reasoning is often easier than division reasoning, CBA-based instruction often emphasizes multiplication over division.

To further illustrate the inverse relationship and how we can replace division with multiplication, suppose students solve the problem 7 × 12 by skip-counting forward: 12, 24, 36, 48, 60, 72, 84. We might then pose the related division problem: How many 12s are in 84? Students might use the same skip-count sequence they used to find 7 × 12 to find 84 ÷ 12. Seven iterations of 12 produced 84, so 84 ÷ 12 = 7. Note that the numbers in this skip-count sequence are the sums resulting from repeatedly adding 12.

Another way to find 84 ÷ 12 is to skip-count by 12 backward from 84—72, 60, 48, 36, 24, 12, 0. Note that the numbers in this backward skip-count sequence are the differences produced by repeatedly subtracting 12 from 84. This is a difficult strategy to use, so CBA doesn’t emphasize it. Because of the inverse relationship between multiplication and division, we can always replace backward skip-counting for division with forward skip-counting in the related multiplication problem.
We might visualize the relationship between multiplication and division in different contexts with the following table:

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>forward skip-counting</th>
<th>repeated addition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Division</td>
<td>backward skip-counting</td>
<td>repeated subtraction</td>
</tr>
</tbody>
</table>

**Starting with Single-Digit Multiplication and Division**

Because students must develop a good deal of proficiency with single-digit (SD) multiplication and division before progressing to multidigit (MD) multiplication and division, in Cognition-Based Assessment (CBA), we first examine levels of sophistication in students’ reasoning about single-digit multiplication and division, along with instructional suggestions for helping students move through these levels. We then examine levels of sophistication in students’ reasoning about multidigit multiplication and division and relevant instruction for this reasoning.

Importantly, before students progress to multidigit multiplication and division, they should develop sufficient fluency with the “basic facts” for multiplication and division (in CBA, the basic multiplication facts are problems that involve products of 2 single-digit numbers). Without adequate fluency with these facts, the cognitive demands required to implement multidigit multiplication and division will be too great for most students to handle. For instance, suppose students attempt to find $6 \times 17$ by multiplying 6 times 10, then 6 times 7, then adding the products. If they have to determine either $6 \times 10$ or $6 \times 7$ by counting by ones or skip-counting instead of quickly recalling either product, they will probably lose track of where they are in the overall computation because the brain’s “working memory” is limited.

Nevertheless, students do not need complete mastery of basic facts before they can make sense of multidigit strategies. Students who know or can quickly derive most of their basic facts can reasonably start exploring multidigit problems. In fact, the distinction between single-digit and multidigit reasoning is not clear cut. For instance, students who regularly use skip-counting to determine products involving single digits often start to use skip-counting for simple multidigit problems like $6 \times 12$.

**Multiplication Before Division**

Although reasoning about multiplication generally develops before reasoning about division, students can often learn both operations at the same time. For example, after students become fluent at one CBA level of multiplication reasoning, they can begin solving division problems requiring that same level of reasoning. Other times, it makes more sense to stick with one operation for several levels. For instance, once students become fluent with multidigit multiplication at Level 3, it makes most sense for students to move next to Level 4 for multidigit multiplication.
Understanding Algorithms

The levels of sophistication in CBA describe students’ development of core concepts and ways of reasoning about multiplication and division. An important part of this development is understanding and becoming fluent with using computational algorithms. However, if algorithms are taught too early in students’ development of reasoning about multiplication and division, students cannot understand the algorithms conceptually, so they learn them by rote. Indeed, most students in traditional instruction learn traditional algorithms for multiplication and division by rote without understanding the underlying number properties. Chapter 2 contains a special section on understanding and determining levels of sophistication in students’ use of computational algorithms.

Understanding Students’ Levels of Sophistication for Multiplication and Division

The CBA approach to teaching students to multiply and divide whole numbers is built around detailed descriptions of levels of reasoning that allow us to tailor our instruction to meet students’ learning needs. At first glance, the amount of detail can be overwhelming. So, keep in mind that understanding CBA levels develops in stages and over time. First, focus on learning the major features of the levels. Then, as you use CBA with your students, you will learn the finer details of the CBA framework.

Zooming Out to Get an Overview

To get an idea of the overall organization of the levels, examine the following “zoomed-out” view of the major ways students think about multiplication and division. Familiarize yourself with these major levels first without worrying about the sublevels that are discussed in Chapter 2.

<table>
<thead>
<tr>
<th>SD Level 0</th>
<th>Student does not understand multiplication and division situations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD Level 1</td>
<td>Student multiplies or divides numbers by counting objects in groups by ones with no skip-counting.</td>
</tr>
<tr>
<td>SD Level 2</td>
<td>Student multiplies or divides numbers by repeated addition/subtraction or skip-counting.</td>
</tr>
<tr>
<td>SD Level 3</td>
<td>Student multiplies or divides numbers by recalling facts or by using properties to derive answers from known facts with no counting or skip-counting.</td>
</tr>
<tr>
<td>MD Level 1</td>
<td>Student multiplies or divides numbers by counting objects in groups by ones with no skip-counting.</td>
</tr>
<tr>
<td>MD Level 2</td>
<td>Student multiplies or divides numbers by repeated addition/subtraction or skip-counting.</td>
</tr>
<tr>
<td>------------</td>
<td>------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>MD Level 3</td>
<td>Student multiplies or divides numbers by using properties to combine or separate parts with no counting or skip-counting.</td>
</tr>
<tr>
<td>MD Level 4</td>
<td>Student uses and understands expanded multiplication and division algorithms.</td>
</tr>
<tr>
<td>MD Level 5</td>
<td>Student uses and understands traditional multiplication and division algorithms.</td>
</tr>
</tbody>
</table>

In the top row of this zoomed-out view, SD Level 0, students do not understand the concepts of multiplication and division. In the bottom two rows, MD Levels 4 and 5, students develop understanding of and fluency with algorithms for multiplication or division, first with expanded algorithms then with traditional algorithms.

In between the top and the bottom rows, students start with counting-based procedures for multiplying and dividing, first by ones then using skip-counting. Then students move to sophisticated, property-based, noncounting procedures that prepare them for deep conceptual understanding of computational algorithms. Roughly the same progression applies, first for single-digit numbers then for multidigit numbers:

**Level 1:** Students treat numbers as collections of ones (e.g., to find $7 \times 5$, the student makes 7 piles of 5 objects and counts them all as ones).

**Level 2:** Students use skip-counting to iterate composite units (e.g., to find $7 \times 5$, the student skip-counts 5 seven times—5, 10, 15, 20, 25, 30, 35).

**Level 3:** Students use known facts and number properties instead of counting (e.g., to find $7 \times 5$, the student reasons that $5 \times 5$ is 25, and $2 \times 5$ is 10, so $7 \times 5$ is $25 + 10 = 35$).

**Zooming In to Meet Individual Students’ Needs**

Understanding individual students’ reasoning precisely enough to maximize their learning or remediate a learning difficulty requires a more detailed picture. We must zoom in to see CBA sublevels (see Figure 1.4). The jumps between sublevels must be small enough that students can achieve them with small amounts of instruction in relatively short periods of time.

Imagine students trying to climb the plateaus in the cognitive terrain described by CBA levels. In situation A, the student has to make a cognitive jump that is too great. In situation B, the student can get from Level 1 to Level 2 by using accessible sublevels as stepping-stones. To provide students with the instructional guidance and cognitive support they need to develop a thorough understanding of mathematical ideas, you need to understand and use the sublevels. Chapter 2 provides detailed descriptions and illustrations of all the CBA levels and sublevels for multiplication and division.
Figure 1.4 Accessible Cognitive Jumps.