YOUNG MATHEMATICIANS AT WORK
Constructing Multiplication and Division

Multiplication and Division Minilessons, Grades 3–5

FACILITATOR'S GUIDE

Antonia Cameron
Carol Mosesson Teig
Sherrin B. Hersch
Catherine Twomey Fosnot

HEINEMANN
Portsmouth, NH

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MATERIALS DEVELOPMENT

Mathematics in the City
City College of New York
Catherine Twomey Fosnot
Director and Principal Investigator

Sherrin B. Hersch, *Co-Principal Investigator*
Antonia Cameron, *Co-Principal Investigator*
Co-Authors, Facilitator's Guides accompanying CD-ROMs

Herbert Seignoret, *Staff Assistant*
Juan Pablo Carvajal, *Staff Assistant*

Freudenthal Institute
Utrecht University, The Netherlands
Maarten Dolk
Co-Principal Investigator

Parul Sleger, *CD-ROM Developer*
Chris Rauws, *Software Developer*
Arian Huisman, *Software Developer*
Han Hermsen, *IT Supervisor*
Logica CMG
Anneleen Post, *Secretarial Support*
Nathalie Kuijpers, *Secretarial Support*
Peter Croeze, Koen Fransen,
Frank Udo, *Interns*

PARTICIPATING TEACHERS

New York City Public Schools

Region 9
Tanya Kessenich, PS 116
Miki Jensen, PS 234
Dana Ostrowsky, PS 234

Region 10
Liza Hernandez, The Muscota New School

City of New Rochelle Public Schools

Carol Mosesson Teig, Jefferson Elementary

FIELD TEST STAFF

Mathematics in the City, City College of New York
Dawn Selnes, *Inservice Director*
Christine Ellrodt, *Inservice Staff*
Nina Lui, *Inservice Staff*
Carol Mosesson Teig, *Inservice Staff*

NATIONAL FIELD TEST PARTNERS

Victoria Bill
Learning Research and Development Center
Institute for Learning, University of Pittsburgh
Cathy Feughlin
The Webster Grove School District
St. Louis, Missouri
Ginger Hanlon, Sarah Ryan, Gary Shevell
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School of Education
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University of California, Santa Barbara
Judit Kerekes
College of Staten Island, New York
Connie Lewis
Tucson, Arizona
Public Schools
Wendy Watkins Thomson
Project Construct, University of Missouri,
Columbia, Missouri

EVALUATION TEAM

Joseph Glick, *Director*
Mara Heppen, *Researcher*
Stephanie Domenici Cabonargi, *Researcher*

VIDEO PRODUCTION

Cathrine Kellison, Roseville Video, *Producer*
Kiyash Monsef, *Co-Producer*
Jeffrey McLaughlin, *Editor*

VIDEO CREW

John Bianchi, Tami Evioni, Richard Henning, John Javakian, Michael Kelly, Anthony McGowan, Serafin Menduina, Mark Petracca, Ben Vandenboom, Richard Westlein

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Overview

Minilessons are typically highly focused and short (ten or fifteen minutes in duration). Their design—a structured series of computation problems or strings—is used to build and highlight number relationships and operations and to develop efficient mental-math computation. To this end, they are guided and explicit in nature.

There are two CD-ROMs in this set of materials on minilessons. One is on multiplication, the other on division. They can be used separately, but it is important to explore with participants the relationship of the operations. For this reason we have packaged them together.

The CD-ROMs give participants opportunities to analyze different kinds of multiplication and division minilessons: strings based on bare number problems and problems that employ contexts (or pictures). In these CD-ROMs, participants are asked to examine each minilesson's design and think about how the choice of number or, in the case of context, how the picture's structure can be used to support the development of children's strategies for computation. Participants are also given opportunities to think about how the open array can be used both as a model to represent students' thinking and as a tool to think with.

As participants work with the CD-ROMs, they will learn to

- listen to and accurately describe students' strategies;
- analyze students' strategies and think about the big ideas underlying these;
- represent students' strategies using a variety of models; and
- design their own minilessons for multiplication and division.

FACILITATION TIP 8

In Activity 2, "76 × 89," participants have been asked to solve the problem 76 × 89 in three ways. Although they will have little difficulty solving this problem using the standard multiplication algorithm, those who have had little experience doing mental math will struggle to solve the problem in a different way.

It is not uncommon in this situation for participants to try a second strategy in which they split the factors into (70 × 80) + (6 × 9). However, when they solve the problem in this way, participants often obtain an incorrect answer because they omit two partial products: 9 × 70 and 80 × 6. In this situation participants, who may not understand how the standard multiplication algorithm works or how their second strategy is related to it, are often confused as to why they have gotten two different answers, but are hard pressed to find the error.

One way to support their understanding of why the standard algorithm works is to have them use graph paper and make an array of the problem. Watch to see if they treat the digits in the algorithm with the appropriate place value. The experience of finding the partial products on graph paper arrays may be the first time participants explore the algorithm with meaning!

At this point as a comparison, it is also powerful to have them explore the algorithm many have learned in algebra: FOIL (first, outer, inner, last) and discuss the connections and the underlying distributive property. Many participants assume that the standard multiplication algorithm is the goal because it will be needed later in algebra. They are surprised to discover that the algorithm taught for whole-number multiplication is the *reverse* of FOIL. Understanding the distributive property is far better preparation for algebra than routinizing a procedure (for an example of this see Appendix F, Dialogue Box C, page 115).

As part of this discussion, be sure to examine other strategies used by participants as well. If these are minimal, you can use some of those described in Chapter 6, "Algorithms Versus Number Sense" in the companion book, *Young Mathematicians at Work: Constructing Multiplication and Division*, page 91. Although you may want to assign this chapter as reading, remember that for those participants on a beginning journey, it might be better to return to this chapter after they have had sufficient experience with doing mental math and using the open array. Doing this will help participants think about their own mental-math strategies and, if they struggled to solve 76 × 89 in more than one way, what in their own mathematical experiences might have contributed to this rigidity of thought. Reflections such as these can have a powerful effect on participants and be catalysts for transforming their practice.

Anticipating Students' Solutions

What solutions do you expect to see for the different problems in this string?

2 × 3

2 × 30

4 × 4

4 × 40

4 × 39

Possible responses you might expect to hear to the question, What strategies do you expect to see? might include: Students will use

- the multiplication facts they know (e.g., 4 × 4 = 16)
- a trick—"add a zero" when problems are related and one of the factors is a multiple of ten (e.g., if 2 × 3 = 6 then 2 × 30 is 2 × 3 "plus" 0)
- the distributive property (e.g., 4 × 39 = (4 × 30) + (4 × 9)
- repeated addition (e.g., 4 × 40 = 40 + 40 + 40 + 40)
- skip counting (e.g., 4 × 40 = 40, 80, 120, 160)
- a doubling strategy (e.g., if 4 × 20 = 80, then 4 × 40 = (4 × 20) × 2)
- the standard multiplication algorithm.

Some participants who do not understand that the word *strategy* implies *mathematical strategy* might focus on the kinds of social interactions and negotiations the students will make as they explore these problems (e.g., "They will use *good* listening strategies.").

Analyzing the Strategies

By clicking on the pull-down list Analyzing the Strategies *you can view Miki Jensen's minilesson. Describe the strategies and big ideas her fourth-grade students are constructing.*

Looking Back

What ideas and strategies are being developed by this minilesson?

In the clips on the CD-ROM page *Analyzing the Strategies*, participants are given an opportunity to view students during the minilesson. Some students' strategies may

ACTIVITY 3 Cross-Cultural Multiplication Algorithms

Explore the following multiplication algorithms from different countries with participants: Egyptian, Russian, South African, Italian (Lattice or Gelosia), and "Cathy's Special" (see Appendix B for examples of these algorithms and analyses of why they work). Begin by doing the algorithms with participants in a rote way (e.g., using one problem, 18 × 22, take participants through each algorithm step by step). Give them a second problem, 46 × 54, to use to "practice" each algorithm. After participants can successfully *do* these different algorithms, ask them to explore why the Russian Peasant and "Cathy's Special" work. Be sure to provide graph paper, which is essential for participants' explorations of these two algorithms. After participants have had time to examine these two algorithms in depth, bring them back together for a whole-group discussion to share their findings.[1] (See Appendix F, Dialogue Boxes F and G, pages 123–128 for examples of facilitators exploring both of these algorithms.)

[1]Occasionally participants think that the purpose of this activity is to teach these new algorithms to their students. If this comes up in discussion, be sure to highlight that the goal here is not on teaching a particular algorithm, but on understanding the mathematical ideas underlying why it works.

ACTIVITY 4 Kid Watching

Use an LCD projector and watch Clip 35 without interruption or discussion. After the first viewing, ask participants to comment on their noticings without commenting on each other's ideas. Record what participants say and look for commonalities and differences. Be sure to highlight any contradictions in their observations and make these the basis for rewatching the clip (for an example of a facilitator doing a *kid-watching* activity, see Appendix F, Dialogue Box H, page 128).

FACILITATION TIP 9

Because it is critical for participants to develop their ability to *kid watch*, before sending them off to work on this page of the CD-ROM, do a *kid-watching* activity with them using 4 × 39. For an example of how one facilitator did kid watching, refer to Appendix F, Dialogue Box H, page 128. The goal of this activity is to help participants hone their powers of observation. In a beginning journey, participants often make interpretive or judgmental kinds of comments (e.g., "The girl seemed frustrated," or "He explained himself clearly."). Be sure to use repeated viewings of the video clip (Clip 35) to help participants learn to accurately describe children's strategies by focusing the conversation on what children are doing and saying (e.g., "Her strategy was to split 39 into 30 and 9 and multiply 30 × 4 and then 9 × 4 and add the products together.").

FACILITATION TIP 10

If you have not done strings with participants prior to their working with this page, expect some of them not to understand how the string will be used to develop computation strategies (e.g., how each problem is presented separately, why different answers are accepted, how strategies are represented, etc.). Do not be surprised if participants view the string as a kind of "worksheet" and think the problems will be presented all at once to the students and solved individually without discussion. They may also not think about the mathematical ideas that can be potentially developed by the juxtaposition of the problems in this string.

FACILITATION TIP 11

As participants work on this page, be sure to go around and look at what they are writing. Some participants may not understand what is meant by a "strategy" and answer the question by writing the *solutions* to the problems in the string (e.g., as $2 \times 3 = 6$; $2 \times 30 = 60$, etc.). If this is the case, you may have to spend some time helping them understand what *strategy* means.

FACILITATION TIP 12

It is not unusual for some participants to wonder why Miki writes the problems horizontally since they have seen computation problems written only vertically. Pose their question back to them and ask how writing the problems as 2×3, 2×30, etc., might influence students' solution strategies.

FACILITATION TIP 13

A common struggle for participants in a beginning journey is knowing how to label the strategies and big ideas they are noticing as they work with the folders on the CD-ROM. For this reason, a glossary is provided on the CD-ROM (see OVERVIEW on the menu bar) as a reference tool to help participants define words. For more in-depth reading on a specific topic, refer them to the companion book, *Young Mathematicians at Work: Constructing Multiplication and Division*.

Nonetheless, it is also important for you to be aware that occasionally participants do focus on finding the "right" label for students' strategies. Because developing participants' powers of observation is so critical in a beginning journey, if you notice this happening it is important to redirect their attention to accurately describing a strategy before looking for the label. You may also find that participants on a beginning journey latch onto one label (e.g., unitizing) and use it indiscriminately. In these instances, you may also have to redirect participants' thinking by highlighting contradictions in their uses of these labels or by referring them to readings in the companion book.

TECH TIP 1

Tools needed by participants to use with *Analyzing the Strategies* are given in the Help file located in the TOOLS menu (see "Viewing Children's Work").

not be surprising (e.g., they know their multiplication facts). Other strategies, however, may have to be explored. For example, participants who solve the problem 2×30 the same way as Gabriella in Clip 32 ("If you add a zero to the 3, then you add a zero to the 6.") may need to explore themselves why this "trick" actually works and why Miki spends time examining this idea with her students. At the heart of this discussion is one of the big ideas underlying Miki's string—the associative property of multiplication. 2×30 can be thought of as $2 \times (3 \times 10)$ or $(2 \times 3) \times 10$.

Another big idea that emerges during Miki's string is the distributive property of multiplication. This occurs when she uses 4×40 and then moves to the next problem in her string, 4×39. Students solve this in two ways; both solutions are based on the distributive property: $4 \times 39 = (4 \times 30) + (4 \times 9)$; and $4 \times 39 = (4 \times 40) - (4 \times 1)$ (see Clips 36 and 40, respectively).

Throughout the minilesson, Miki uses an array to model students' strategies. She progressively develops the open array model, starting with graph paper, which is used to model the connection between the first two problems in her string, 2×3 and 2×30. In this discussion, Miki explores the associative property of multiplication by highlighting how many times the 2×3 array is contained within the 2×30 array. As she marks the 2×3 array on the 2×30 array, students count how many times it "fits." It fits 10 times because 2×30 can be thought of as $(2 \times 3) \times 10$.

FACILITATION TIP 14

Many participants may know the zero "trick" in multiplication, but struggle to explain why it works. Because they may have been taught mathematics as a series of tricks to be learned and applied, it is important in these discussions for you to highlight how the mathematics is not in being able to do the "trick," but in knowing why the trick works.

You might want to explore the zero "trick" with them by putting the problem 10×33 on the board and asking them to solve it. Many participants will say they know the answer is 330 because you can just "add" or "put" the zero. That this trick works because of the commutative property of multiplication, that $10 \times 33 = 33 \times 10$, will be something that you will have to explore in discussion. To see how one facilitator handled this discussion, see Appendix F, Dialogue Box A, page 112.

FACILITATION TIP 15

While participants may discern how Miki's string brings the distributive property up for discussion, they may not recognize how she is also using her string to develop the idea that some uses of the distributive property are more efficient than others. Because the development of computational efficiency is so critical to understanding one of the big goals behind doing mental math with students, it needs to be explored with participants. To do this, you might want to examine the two ways Miki's students use the distributive property to solve 4×39: First, $(4 \times 30) + (4 \times 9)$; and second, $(4 \times 40) - (4 \times 1)$. Their first strategy is related to the standard multiplication algorithm, which, when used with more complex problems, can lead to computation errors. Compare the following solutions to the problem 19×36. Both are based on the distributive property of multiplication. Solving the problem by changing it into a related problem (e.g., $19 \times 36 = (20 \times 36) - (1 \times 36)$) is more elegant and less cumbersome than using a strategy based on finding the partial products (e.g., $19 \times 36 = (9 \times 6) + (9 \times 30) + (10 \times 6) + (10 \times 30)$), which places a heavy burden on memory. Thus, while the distributive property is an important mathematical idea for students to understand, knowing *how* to use the distributive property efficiently in mental math requires a different kind of understanding, one that is related to the development of key number relationships.

FACILITATION TIP 16

As participants work with the string in this folder, they may comment on Miki's use of the array to represent student strategies. Because participants may be unfamiliar with this model and the ways in which it can be used to develop and support students' thinking, some discussions may have to focus on how Miki is representing student strategies. However, it is important to remember that the open array will be explored more in depth in *Journey 2*. Here a significant part of participants' work will be to examine the different ways the array is used in the minilessons in *Journey 1: Multiplication and Division*, and think about how it is used as a model to represent and support thinking.

FACILITATION TIP 17

Expect participants who have never seen or experienced a string to be struck by Miki's pedagogy. Although pedagogy will be examined in depth in *Journey 2*, participants at this stage of their work may also need to have some conversations about (1) how Miki structures her discussion; (2) the kinds of questions she uses; (3) how, when, and why she explores a student's strategy (e.g., why she might spend time exploring a student's strategy for 4×40, but not spend time on 4×4); and (4) her choices for representing students' thinking.

In the other segment of her string, Miki continues to explore the distributive property of multiplication, but now moves away from modeling student strategies with an array made from graph paper to drawing an open array to represent their thinking. Because Miki recognizes that students' strategies for 4×39 may be to remove one group of four from 4×40 (e.g., $4 \times 39 = (4 \times 40) - (4 \times 1)$), she draws an open array for 4×40. Miki uses the open array not only to model students' strategies, but also to build connections between them.

What Would You Do Next?

◎ *If you were Miki, what string would you follow this lesson with, given the strategies you saw the children using?*

FACILITATION TIP 18

Keep participants' answers to the question on this page grounded within the work they have just done analyzing Miki's string. If you find participants losing focus or beginning to think about things they would do in their own practice that have nothing to do with mental math (e.g., giving students the strategy they would like them to know and having them practice this in a rote way), review Miki's string with them with an emphasis on her choice of numbers and how these choices affected the children's strategies. It may also be helpful to participants if you remind them that the problems in a string should be related in ways that will support the development of specific strategies.

It may be difficult for participants on a beginning journey to imagine what Miki would do next. For this reason, it is critical that you clearly delineate what your expectations are for them as they work on this page. It might be helpful to begin by reexamining the structure of Miki's string and asking participants to reflect on her choice of numbers and the ways in which the problems were scaffolded to support students' strategies. Although other aspects of her string may come up for discussion—the questions she uses; the *rhythm* of her string (when she slowly allows for exploration and when she moves quickly to the next problem); and how she models students' strategies on the array—keep the conversation focused on designing the next problem so that it builds upon and works with students' strategies. This will be sufficiently difficult for many participants at this point in their journeys.

Some of the problems participants design as possible next steps may vary greatly. Keep in mind that what they create will be connected to their understanding of

- the landscape of learning for multiplication (big ideas, strategies, and models);
- how students develop multiplication strategies and the bridges between additive and multiplicative reasoning;
- how the choice of numbers in a given string has the potential to *suggest* certain strategies (e.g., in the pairing of the problems 4×40 and 4×39 one potentially realized suggestion was "using what you know"—the answer to 4×40—and adjusting this product by subtracting one group of 40); and
- how the choice of problems and the ways in which they are sequenced have the potential to shift student strategies and develop computational efficiency.

Backburner: A Moment for Further Reflection

◎ *This is the last page in the folder "Strings: An Example." However, you may have other questions on this topic that you would like to investigate. Go to the TOOLS menu above and add them to your* Backburner *notes.*

The *Backburner* page is a tool for *both* participants and facilitators. As participants work with the materials, they may raise many questions that may not be easily or immediately answered. The *Backburner* page offers them a place to keep their questions for another time. These may be answered or evolve as they work more deeply with the CD-ROM.

340 ÷ 17

◎ *By clicking on the images below, you can see Carol Teig, a fifth-grade teacher in New Rochelle, New York, in the midst of a minilesson. Her minilesson is comprised of a string of related problems. The first three problems are 340 ÷ 17; 68 ÷ 17; 408 ÷ 17. In the clips below, you see Carol introducing the problem and Christina sharing her solution to the first problem. What strategy is she using?*

To solve 340 ÷ 17, Christina (Clip 304) uses a landmark ten-times strategy. Implicit in Christina's strategy are two big ideas: the distributive property of multiplication (10 × 17) + (10 × 17) = 20 × 17; and the connection between multiplication and division—340 ÷ 17 can be thought of as 17 × ? = 340.

While it may be relatively easy for participants to delineate Christina's strategy, you may have to spend time exploring the big ideas underlying this strategy with them. For example, Christina is able to use her understanding of part-whole relationships (she keeps the dividend in mind in her strategy: (10 × 17) + ? = 340) and the idea that division is the inverse of multiplication to help her solve the problem.

FACILITATION TIP 33

As participants examine students' strategies in this folder, it is important for them to recognize that multiplying by 10 is an important strategy for students to have in their computational repertoire. Developing this strategy is a key step in students' mathematical development because understanding what happens to a number when it is multiplied by 10 (or 100 or 1,000) is a big idea in mathematics. The development of this idea has a tremendous impact on children's strategies in multiplication and division.

FACILITATION TIP 34

At this point in participants' journeys, where they may still be developing their ability to *kid watch*, it may be more helpful for them just to focus on learning to accurately describe students' strategies. Since novices in a beginning journey often jump quickly to assigning labels to students and strategies, it will be helpful to them if you discourage these kinds of behaviors. Their ability to distinguish between strategies, big ideas, and models will develop as they continue to work with the CD-ROM and read the companion book, *Young Mathematicians at Work: Constructing Multiplication and Division*.

Related Problems

◎ *Click on the pull-down list* Related Problems *to view more of this string. What strategies do the students use? What mathematical ideas are they constructing? Make notes for both.*

FACILITATION TIP 35

The ratio table (see the list of potential student strategies on page 19) is an important mathematical model. Here it is used as a tool to organize computation. The strategy shown, however, is but *one* possible way students might represent their solutions to 408 ÷ 17 using a ratio table. For more information on the ratio table, see Background Readings located under INFO on the menu bar of the CD-ROM.

68 ÷ 17

Some students' strategies may not be surprising or difficult to understand. For example, Nicole (Clip 315) uses repeated addition to solve 68 ÷ 17, saying "I did 1 by 17, 2 by 17, 3 by 17, and 4 by 17" and then, after adding up her partial products, knows that "17 and 17 was 34, and 34 plus 17 was 51 and one more 17 was 68." Other strategies, however, may not be as easy for participants to understand and, therefore, may have to be explored with them in whole-group discussions.

In Clip 318, Michael uses his knowledge of 17 × 10 to figure out that 17 × 5 would be 85, and says, "because 85 is half of 170." He then subtracts one group of 17 from 85 to obtain the dividend of 68, and gets his answer, "4" by subtracting "1" from "5."

Michael's strategy reveals a flexible use of number relationships. Because participants themselves may not have developed a rich network of number relationships, expect some to be confused by his strategy or consider it highly inefficient. Participants may feel Michael's strategy is too convoluted for such a "simple division problem." After all, 68 ÷ 17 can easily be solved with a doubling strategy: 2 × 17 = 34, 34 is half of 68, so 4 × 17 = 68 (see Jackie's strategy, Clip 310).

To understand the number relationships Michael employs in his strategy, participants need to consider (1) why does Michael begin with 17 × 10 when its product is obviously not close to 68, the dividend in the given problem? (2) What kinds of mathematical ideas must Michael have constructed about number relationships to use such a strategy?

One way to help participants think about Michael's strategy is to compare it with those used by other classmates and to think about what these strategies have in common and how they are different. For example, how is his strategy similar and/or different from Jennifer's?

Both he and Jennifer (Clip 311) use a ten-times strategy. Both he and Jennifer realize that 170 was "too much" and that they need to remove groups of 17. The difference is in how they remove these groups of 17. Jennifer subtracts groups of 17 from 170. This is a cumbersome strategy that she abandons, saying, "I took away, but I thought that would be too much [*sic*] steps so then I stopped." Michael, however, uses a shortcut strategy connected to an important mathematical relationship: $n \times 5 = (n \times 10) \div 2$. When he cuts the 170 in half, the 85 he obtains is much closer to the dividend, 68, in the original problem. This flexible strategy use—the adjusting and playing with numbers to create elegant and efficient solutions—is a hallmark of number sense.

FACILITATION TIP 36

Carol's string is designed to bring the distributive property of multiplication up for discussion. However, not all uses of the distributive property are efficient. As you examine the clips with participants, be sure to bring up the idea of what makes a strategy efficient.

In Clip 315, this happens naturally as part of the minilesson when Jackie comments on Nicole's repeated addition strategy: "She already knew 17 + 17 was 34. Why didn't she just put on another 34 to just make it 68?" Have participants think about how Carol handles this part of the discussion and why she might have done this. You might also ask participants to find other examples of how this notion of efficiency is underscored by Carol in ways that support the idea that while all strategies may be accepted, not all strategies are equally efficient.

FACILITATION TIP 37

Participants need to consider how the structuring of the problems in Carol's string might have affected students' strategies. Since this discussion is rather sophisticated (the juxtaposition of problems for didactical purposes), you may have to support participants by pointing out a simple observation that you (or another participant) had made as you were analyzing the clips: "Three of the four children (Jackie, Michael, and Jennifer) who share their solutions to the problem $68 \div 17$ use 10×17 as a starting place." Ask participants to think about (1) Why might this have happened? (2) Was this coincidental or did the problems in Carol's string somehow affect students' strategies?

Although participants may begin to think about how the initial problem, $340 \div 17$, supported the use of ten-times strategies (e.g., 10×17 or 20×17), they may not think there is any connection between that problem and the next one. One detail participants may overlook is that the stretch for many students with $68 \div 17$ was not in solving it, but in *how* to use the ten-times strategy (if they chose to begin there) effectively. The application of the ten-times strategy was problematic, but it also had the potential to stretch student thinking. Michael's shortcut—dividing the product of 10×17 in half—was one solution to the dilemma of using a ten-times strategy to solve $68 \div 17$.

408 ÷ 17

Although Carol's students have been using the distributive property to solve the problems in the string, the next problem, $408 \div 17$, is built into the string to push students toward using this landmark strategy. As students think about how to mentally solve such a messy problem, they can play with the problem using the distributive property to help make it easier to solve mentally.

One possible solution is to change $408 \div 17$ into $(340 \div 17) + (68 \div 17)$. While Carol's string makes this explicit by its design, participants need to think about how, over time, and as students internalize important strategies and develop a rich network of mathematical relationships, using this strategy will not be dependent on a "string." The string students now create will be in their own minds as they look to the numbers in a given problem to decide on the best—most elegant and efficient—solution!

As participants observe Carol's string they may wonder when and how students actually master specific strategies. Those who hold a behaviorist approach to learning, where students' mastery of a skill (in this case the desired strategy) is based on a considerable amount of practice, may wonder why Carol does not tell students which strategy is best and encourage them to try to use this strategy. If this idea comes up in discussion, explore with participants the pros and cons of letting students generate their own solutions.

The following questions might help participants think about the benefits of doing mental-math minilessons: Why does Carol give students an opportunity to solve the problems in their own ways? What potential benefits might this have on students' mathematical growth and development? What evidence can you find in the video clips to support how this minilesson is affecting student thinking?

As participants cite reasons for more direct kinds of teaching, remember that some of what they say might be rooted in their own fears about how they would implement minilessons in their own teaching. These can include issues around (1) losing control of their students (e.g., "My students would never sit that long without a fight breaking out."); (2) not knowing how to create a community of discourse (e.g., "I could never do that with my students; they would never listen to each other that way."); (3) being unsure of how students learn if they are not directly taught (e.g., "The only way I know to get all kids to use a ten-times strategy is to have them practice it and practice it some more."); and (4) having (and being aware of) their own limited content knowledge (e.g., "I found some of the strategies that students shared so confusing. I wondered, if I were Carol, how would I know what kinds of questions to ask?").

There is no doubt that for some participants, showing students a strategy and having them practice it is closely aligned to what they do in their own practice. Trying to listen to and understand a strategy that may be confusing or complicated or that may bring into focus the limitations of their own content knowledge is a scary place for many teachers to be. No matter what comments come up in this type of discussion, remember that it is helpful for participants to have a forum to articulate their fears so that they can begin to think about ways to move beyond them and change their practice.

Looking Back

◎ *What ideas and strategies are being developed by this minilesson? If you were Carol, what problems would you follow with? Make notes on both questions.*

Participants, by this point in their work, have had ample opportunity to analyze the students' strategies and think about the mathematical ideas they were using and/or constructing. Although you may have explored some of the students' strategies together in whole-group discussions, you may not have had many opportunities to assess how participants integrated these ideas into their own analyses or how clearly they understand the mathematical ideas in students' strategies and the ways in which different strategies are connected.

This last section of the page *Related Problems* can be used in a variety of ways to help you gain insights into participants' understanding not only of the different kinds of strategies, but what these strategies mean mathematically. As you consider the kinds of activities you may need to do with participants to deepen or strengthen their content knowledge, it might be a good idea to give this page as a written assignment so that you can use participants' analyses as the basis for your next teaching steps.

Last Two Problems

◎ *Carol's last two problems are 323 ÷ 17 and 399 ÷ 21. Given what you have seen the students do so far, what strategies do you expect the students to use here?*

Because participants have had experiences watching and analyzing Carol's students' strategies, their expectations for students' solutions to the problem 323 ÷ 17 will more than likely be connected to previous strategies they have analyzed. These might include students using

- prior information, such as
    - taking the problem 340 ÷ 17 = 20 and subtracting 17 (e.g., 323 = 340 – 17; if 17 × 20 = 340, then 17 × (20 – 1) = (340 – 17))

    ◦ putting together the areas from the previous problems that would equal 323 (e.g., (170 + 85 + 68 = (17 × 10) + (17 × 5) + (17 × 4)), and using this information, 10 + 5 + 4, to find the quotient to 323 ÷ 17)
- ten-times strategies and counting up or back by 17s (e.g., 17 × 10 = 170; 170 + 17 = 187; 187 + 17 = 204, etc. or 17 × 20 = 340, so 17 × 19 = 17 × (20 – 1) = 323);
- doubling strategies (e.g., 2 × 17 = 34; 4 × 17 = 68; 8 × 17 = 136, etc.).

Although it may not be difficult for participants to predict students' strategies for 323 ÷ 17, they may struggle to understand the connection between this problem and the next one and think that perhaps there is some mistake in the string's design. Their predictions for students' strategies for 399 ÷ 21, therefore, may be quite varied and not connected to what has occurred previously in the minilesson.

The last problem in the string is not a fluke, but has been carefully built into the string's design to challenge students to apply some of the strategies that have come up during the lesson. How the choice of numbers *suggests* using a ten-times strategy and an efficient use of the distributive property needs to be explored in whole-group discussions of this CD-ROM page. The following questions might be helpful in facilitating such a discussion:

- How do these last two problems in the string build upon the mathematical ideas that have been previously discussed by Carol's students?
- How might students use the previous problems in the string to efficiently solve 323 ÷ 17? How might this strategy then affect their solutions for 399 ÷ 21?
- What is the relationship between 399 and 21 that can be used by students as they consider strategies for solving this division problem? How might this choice of numbers support the use of landmark strategies?

To help participants understand how the problems in the string are scaffolded to support both a ten-times strategy and an efficient use of the distributive property, it is important to examine how the problems in the string are similar and how they are connected to each other. In the last two problems the dividends have been carefully chosen to be about 20 times the divisor. Because 323 = 340 – 17, students can use this relationship to solve 323 ÷ 17. If 340 ÷ 17 = 20 (a problem previously solved), then 323 ÷ 17 = 20 – 1. The suggestion built into the string is to use a similar kind of strategy to solve 399 ÷ 21. If 420 ÷ 21 = 20, then 399 ÷ 21 = 19 because 420 ÷ 21 = (399 + 21) ÷ 21 = 19 + 1.

But there is another important relationship suggested by Carol's choice of numbers that needs to be highlighted in whole-group discussions. This is that the numbers in the last problem clearly suggest the use of landmarks. Here 399 is almost 400 and 21 is almost 20. Students can use this information to estimate that the quotient of 399 ÷ 21 will be about 20 by using landmarks (e.g., if 20 × 20 = 400, then 21 × 20 *is a little more than 400*).

323 ÷ 17

◎ *By clicking on the images below, you can see the students discussing 323 ÷ 17. What strategies do the students use? What mathematical ideas are they constructing? Make notes for both.*

In the clips on this page, both Ernie and Jackie use the distributive property of multiplication to solve the problem, 323 ÷ 17. In Clip 330, Ernie is able to think of the previous problem 340 ÷ 17 and then just take one group of 17 away from the dividend 340 to get the new dividend 323, thus making the quotient 19, as he says "one minus 20 is 19." In Clip 335, Jackie uses a ten-times strategy, (10 × 17) + (10 × 17) and removes 1 × 17 to find the quotient to 323 ÷ 17.

In order to draw a connection between the two strategies presented to the class, Carol asks the students how Jackie's strategy can be represented on the array she has drawn to model Ernie's strategy. Here two important ideas come up for discussion: first, in connecting how $20 \times 17 = (10 \times 17) + (10 \times 17)$, Carol is highlighting the distributive property of multiplication; and second, in asking the class, "Are these two strategies similar?" Carol is also pushing students to think about computational efficiency. Jackie's response, "Yeah, except like using one more step." touches on this—sometimes the more elegant and efficient strategy uses fewer steps. This idea is an important one for students to develop if they are to become more adept at solving computation problems mentally.

399 ÷ 21

◎ *On this page, you can view the students discussing 399 ÷ 21. What strategies do the students use? What mathematical ideas are they constructing? Make notes for both.*

This problem in Carol's string stretches her students to apply some of the strategies and ideas they have worked on in the previous problems. That Carol expects this problem in her string to be problematic is reflected in how she begins this discussion by writing down the different answers students offer and posing a question, "Is there an answer up here that you say definitely can't be the right answer?"

This discussion emphasizes two critical components of number sense: (1) what constitutes a "reasonable" answer to a given problem; and (2) how one can look to the numbers before solving the problem to determine what a reasonable answer might be. In this regard, "playing" with the numbers before actually solving the problem (e.g., "399 is about 400 and 21 is about 20, so the quotient has to be about 20") can help one estimate the answer.

In the beginning of Clip 338, the initial discussion focuses on students talking about the "reasonableness" of their peers' answers. Here, James says that neither 23 nor 24 are reasonable answers because "21 times 20 is already 420, and that's more than 399." After students discuss and agree that $21 \times 20 = 420$, James continues, "If it's 21×23 that would be even more than 420 and that would be even further away from the target number that's 399." Using this reasoning, Ashleigh realizes that the solution then has to be less than 20.

When Jackie still thinks part of her answer will be correct ("It [the answer] would have to have remainders."), Carol explores her thinking with the class. This segment of Clip 338 needs to be examined with participants in a whole-group discussion because there are a number of important ideas that come up here.

First, Carol explores "wrong" thinking, by asking Jackie to explain her strategy. This might be a novel idea to some participants, who place an emphasis on getting an answer as opposed to exploring the thinking behind an answer. It will be helpful for participants to think about why exploring both incorrect and correct answers might be equally valuable to students.

Second, in this clip there is an example of how doing mental math with students affects their thinking: this occurs when Jackie uses Michael's halving strategy as part of her solution (e.g., $21 \times 10 = 210$, so $(21 \times 10) \div 2 = 210 \div 2$). Thus, using mental math with students has the potential to change student strategies; exploring incorrect solutions has as much power to deepen student thinking as exploring correct ones.

As the class explores Jackie's strategy, she finds a computation error (she added the partial products 210 + 105 + 105 incorrectly and obtained 320 instead of 420). When she realizes this error, Jackie corrects her solution to include the information that $20 \times 21 = 420$, so "then you have to minus 1 (from the 20) because you have to minus one group of 21."

Carol uses the array model to draw connections between the different strategies. She does this when she connects Michelle's strategy: $(21 \times 10) + (21 \times 10) - (21 \times 1)$ to Jackie's strategy that has been modeled on the open array. Here Carol asks, "Is another 21 10 times on this picture?" This question not only helps students to connect how (21 × 5) + (21 × 5) = 21 × 10, but also which strategy might be more efficient. Jackie responds, "So she [Michelle] did fewer steps."

Throughout her minilesson, Carol has been honing students' computational strategies. One technique is to highlight connections between the strategies and model these on the open array. One idea that arose during this minilesson is that of computational efficiency. Although Carol does not explicitly state how one strategy is more effective than another, it naturally became part of the discussion as students thought about each other's ideas. Underlying these discussions is the idea that what makes a strategy efficient is how many steps are contained in the solution.

But what makes a solution efficient? This discussion is important to have with participants to help them understand that what Carol's students are grappling with goes to the heart of what it means to have number sense. The search for elegant, efficient strategies is a quest for simplicity. A critical part of this journey is the development of a network of number relationships—one primary reason for doing mental math strings with students in the first place!

FACILITATION TIP 38

One possible strategy that participants may discuss is how students use an easy multiplication fact, 21 × 2, to know a more difficult one, 21 × 20. In some instances this might be described as "knowing the trick"—when you multiply any number times ten, you just "add" or "put the zero." If this comes up in conversation, be sure to explore why this "trick" works if you have not already done so previously. For more information on this "trick" and why it works, see Facilitation Tip 12 on page 6.

FACILITATION TIP 39

During Carol's string students have been commenting on efficiency by noting and comparing the "number of steps" in a given strategy. But what makes a strategy efficient? Participants now have had enough experience examining and discussing students' strategies in the folder "Strings: An Example" to be able to give examples of what they think is an efficient strategy.

In your discussions with participants on efficiency, do they, for example, think that what is efficient is dependent on the student? Do they recognize that Carol is pushing her students to generalize about strategies and that, depending on the numbers in a given problem, one strategy might be more efficient than others? How do they determine what makes an efficient strategy? Can they articulate, for example, the idea that using a landmark ten-times strategy is an efficient way to work with part-whole relationships in division problems, but that simplifying a problem can also lead to elegant solutions? For example, 340 ÷ 17 can be solved by using the distributive property (10 × 17 = 170; 10 × 17 = 170; 170 + 170 = 340, therefore 340 ÷ 17 = 20) or the associative property (2 × 17 = 34; therefore 20 × 17 = 340 because 20 × 17 = 10 (2 × 17)), but the latter solution, with fewer steps, is less taxing on the memory, and might be considered more efficient.

If participants struggle to define or give examples of efficient solutions, it might be helpful to have them return to the CD-ROM and review different students' strategies for one problem (e.g., 323 ÷ 17). Have them think about how these strategies are related and if some strategies are shortcut versions of others.

Backburner: A Moment for Further Reflection

This is the last page in the folder "Strings: An Example." However, you may have other questions on this topic that you would like to investigate. Go to the TOOLS menu above and add them to your Backburner *notes.*

See *Backburner* notes, page 8.

ACTIVITY 9 **Finish the Strategy**

Give participants the strategy below and ask them to finish it based on the logic/mathematical ideas shown thus far. Ask participants to think about: Will this strategy work? Why or why not? What mathematical ideas is the student using?

Problem	*Student's strategy*		17
368 ÷ 17	10 × 17 = 170 10 × 17 = 170	10	170
		10	170

A SECOND EXAMPLE

300 ÷ 12

How would you solve 300 ÷ 12 = ?

Give participants time to solve the problem on their own and, if they use the long division algorithm, encourage them to try a different strategy. Have a whole-group discussion in which you chart participants'