The Fostering Geometric Thinking Toolkit

The Fostering Geometric Thinking Toolkit

A Guide for Staff Development

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Facilitation Notes

DVD-ROM containing session handouts, video clips, sample student work, computer applets, and slideshow presentations

About the Authors

Mark Driscoll is the author of the Heinemann titles *Fostering Geometric Thinking* as well as *Fostering Algebraic Thinking* and its professional development companion, *The Fostering Algebraic Thinking Toolkit*. Through Heinemann Professional Development he presents further PD related to math teaching and learning. Mark is codirector of the Center for Leadership and Learning Communities at Education Development Center, Inc., where he has directed a range of teacher enhancement, leadership, and materials-development programs.

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Introduction I

Orientation to the Materials

Overview

These professional development materials are a series of group study guides focused on geometric thinking, intended for use by grades 5–10 mathematics teachers in a professional development setting. Twenty two-hour sessions combine to produce forty hours of meeting time and offer the following:

- a conceptual framework to help teachers understand middle school students' thinking in geometry and measurement and to guide them in engaging their students' thinking more productively
- hands-on investigation of rich mathematical problems in geometry and measurement and tools for discussion and reflection aimed at deepening teachers' understanding of geometric thinking
- structured approaches to gathering and analyzing data about how students' thinking about geometry and measurement develops
- structured approaches to discussion among teachers about mathematics, curriculum, student thinking, and other issues related to teachers' practice[1]

Goals

The sessions of *The Fostering Geometric Thinking Toolkit* ("*Toolkit*") provide teachers with challenging mathematics problems and prompt them to analyze artifacts of student thinking. As teachers work

[1]For facilitators who are wondering what it takes to lead this professional development, more information is provided in the Facilitation Guidelines and Organizing Your Group sections.

together, they are encouraged to reflect on their understanding of geometry and the nature of geometric thinking. The major goals of the *Toolkit* sessions are to:

1. Strengthen Teachers' Understanding of Geometry By

- providing opportunities to engage in geometric thinking
- adding to the depth and breadth of their geometric knowledge
- promoting connections among different topics within geometry
- promoting connections between geometry and other areas of mathematics

These materials devote a series of sessions to each of three major content areas in middle-grades geometry: properties of geometric objects, geometric transformations, and measurement of geometric objects. Several sessions also combine geometric topics often kept separate in mathematics curricula (e.g., properties and dissections, transformations and area, etc.) and incorporate mathematical topics often kept separate from geometry (e.g., rational number and algebra). Structuring the professional development as such allows teachers to gain a deeper understanding of geometry and its relevance to other areas of mathematics.

2. Enhance Teachers' Capacity to Recognize and Describe Geometric Thinking By

- providing them with a framework for conceptualizing their own geometric thinking
- providing them with a framework for conceptualizing the geometric thinking of their students

These materials introduce a framework for conceptualizing important types of geometric thinking. This framework, termed the *Geometric Habits of Mind* (*GHOMs*), provides teachers with language to describe geometric thinking and a lens through which to view and analyze their work and the work of their colleagues and students.

3. Increase Teachers' Attention to Students' Thinking By

- encouraging teachers to rely heavily on evidence when making assertions about student thinking
- emphasizing attention to geometric thinking in accurate, as well as inaccurate or incomplete, answers
- equipping teachers with an improved ability to expose and explore students' thinking

The materials model questioning aimed at exposing and exploring thinking and provide teachers with opportunities to brainstorm together about questions they might ask students. The accuracy of a students' answer only hints at their thought processes. Therefore, these materials encourage teachers to analyze students' work for patterns of thinking and then to reflect on whether that thinking is productive for geometric problem solving. Teachers are prompted to provide evidence for any inferences they make in an effort to ensure analyses of student thinking are solidly based in students' work.

4. Enhance Teachers' Understanding of Students' Geometric Thinking By

- improving teachers' capacity to anticipate students' patterns of thinking and to see the potential for advancement in those patterns of thinking
- helping teachers better appraise and respond to students' unexpected methods as well as difficulties in conceptual understanding

The *Toolkit* materials provide teachers with multiple experiences analyzing student work. In addition, teachers will read summaries of relevant research that recount the development of students' thinking about particular geometric ideas. These two components of the professional development give teachers insight into the development of geometric thinking and how that thinking can go awry.

5. Prepare Teachers to Advance Students' Geometric Thinking By

- educating teachers about how to pose problems that maximize potential for students' geometric thinking
- encouraging teachers to pose questions that expose and enhance students' geometric thinking
- instilling in teachers the importance of having students use mathematical arguments instead of procedural explanations when describing problem solving

As the teachers work on, reflect on, and then discuss the mathematics they do together, questions in the materials prompt them to reflect on their use of GHOMs and to go beyond explaining procedures to rely on mathematical arguments when describing problem solving. To help teachers promote the same ways of thinking in the classroom, the materials provide guidelines on how to adapt problems such that they provide the greatest opportunity for students to employ GHOMs and exercise mathematical argumentation.

How *The Fostering Geometric Thinking Toolkit* Works to Approach Goals

A guiding premise of our work is that good mathematics teaching begins with understanding how mathematics is learned. Our intent in these materials is to develop teachers' attention to students' ways of thinking about geometry through the exploration of teachers' own geometric thinking, that of their colleagues, and that of their students. Three guiding structures are in place in these materials to help teachers meet the goals described above.

- The Structured Exploration Process guides the activities in each pair of *Toolkit* sessions, providing a meaningful cycle of exploring and reflecting on mathematics together and exploring and reflecting on student thinking together.
- The GHOMs framework provides a lens for teachers to use when analyzing their own geometric thinking, colleagues' geometric thinking, and students' geometric thinking.
- Three content strands (properties of geometric objects, geometric transformations, and measurement of geometric objects) divide the sessions into sets focused on different important areas of geometry and measurement. (See the next section, Links to Classroom Content, for more information about this guiding structure.)

The Structured Exploration Process[2]

The Structured Exploration Process is an essential component of these materials. The process allows teachers to see mathematics from different points of view and gain a deeper understanding of geometry. The Structured Exploration Process is a cyclical process that repeats each time teachers engage with a new mathematics problem. The cycle involves five stages:

Stage 1: Doing mathematics. Teachers work together with colleagues to explore and solve mathematics problems they will later use with their students.

Stage 2: Reflecting on the mathematics. Using an explicit conceptual framework (the GHOMs), teachers discuss the mathematical ideas and their thinking about the problem.

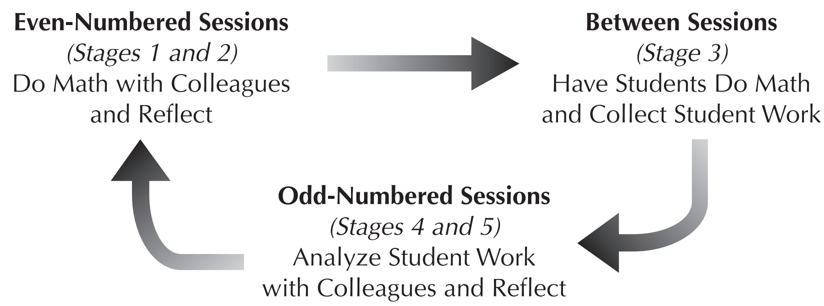
Stage 3: Collecting student work. Teachers use the problems in their own classes and collect student work.

Stage 4: Analyzing student work. Teachers bring selected student work back to the study group to analyze and discuss with colleagues.

Stage 5: Reflecting on students' thinking. Once again using the GHOMs framework, teachers discuss students' mathematical thinking, as revealed in the student work, and ways to elicit more productive thinking in future classes.

[2]Kelemanik, G., Janssen, S., Miller, B., and Ransick, K. 1997. *Structured Exploration: New Perspectives on Mathematics Professional Development*. Newton, MA: Education Development Center.

We believe that, over time, this repeated process leads to clearer understanding of geometric thinking. In the *Toolkit* materials, teachers will engage with eleven different geometry problems. Nine of these problems are used as the medium for working through these five stages (the other two problems will only be used to engage with stages 1 and 2). For each problem, the cycle starts with the work during an even-numbered session, continues between sessions, and is completed during the subsequent odd-numbered session.



The Geometric Habits of Mind

Why Geometric Habits of Mind?

Mathematical habits of mind are productive ways of thinking that support the learning and application of formal mathematics. A major premise of these materials, drawn from our previous work[3], is that the learning of mathematics is as much about developing these habits of mind as it is about understanding established results in the discipline called *mathematics.* Further, we believe that the learning of formal mathematics need not *precede* the development of such habits of mind. Quite the opposite is the case, namely, that developing productive ways of thinking is an integral part of the learning of formal mathematics.

In this light, learning geometry involves learning to think geometrically. In a 1982 address, the great mathematician Sir Michael Atiyah put geometric thinking into a broader perspective:

> Broadly speaking I want to suggest that geometry is that part of mathematics in which visual thought is dominant whereas algebra is that part in which sequential thought is dominant. This dichotomy is perhaps better conveyed by the words "insight" versus "rigour" and both play an essential role in real mathematical problems.

[3]Cuoco, A., Goldenberg, E., and Mark, J. 1997. "Habits of Mind: An Organizing Principle for Mathematics Curriculum" *Journal of Mathematical Behavior*, 15(4): 375–402.

Driscoll, M. 1999. *Fostering Algebraic Thinking: A Guide for Teachers Grades 6–10.* Portsmouth, NH: Heinemann.

Driscoll, M., with Goldsmith, L., Hammerman, J., Zawojewski, J., Humez, A., and Nikula, J. 2001. *The Fostering Algebraic Thinking Toolkit*. Portsmouth, NH: Heinemann.

> The educational implications of this are clear. We should aim to cultivate and develop both modes of thought. It is a mistake to overemphasise one at the expense of the other and I suspect that geometry has been suffering in recent years.[4]

We believe that when people use insight and rigor to explore and solve geometry problems, certain habits of thinking come into play. We also believe that instruction can be shaped to foster the development of such habits of mind in students. That is the core principle of the *Toolkit* materials.

For these materials, we have put together a GHOM "framework" including several habits that seem to be critical to developing power in geometric thinking. The list isn't meant to be comprehensive. However, by learning to attend to these several habits—in their own, in their colleagues', and in their students' work—teachers can become better prepared to help students succeed in geometry.

Selecting GHOMs for our framework has been an extended as well as iterative process, with revisions driven by several forces: conversations with project advisors (both mathematicians and mathematics educators) and with pilot and field-test teachers; examinations of the research literature on geometric thinking; and analyses of artifacts of student work on the problems that have been used in our pilot and field tests. Throughout, we have been guided by four criteria.

1. *Each GHOM should represent mathematically important thinking.* When people who use geometric thinking regularly look at our list, we want there to be no doubt that the habits of mind we emphasize are important to emphasize. Although we do not claim to be comprehensive—that our set of GHOMs encompasses *all* important geometric thinking—we do want consensus that we have chosen important lines of geometric thinking, particularly as they contribute to geometric problem solving.
2. *Each GHOM should connect to the research literature on the learning of geometry and the development of geometric thinking.* Given that we want to emphasize important lines of geometric thinking, we also want to point teachers toward insights gained by researchers into the development of such thinking as well as insights into common hurdles faced by learners in that development.
3. *Evidence of each GHOM should appear often in our pilot and field-test work.* We want to ensure that the lines of geometric thinking we choose to emphasize will show up, with some frequency, in the work of students in grades 5–10, even if the appearance may

[4]Atiyah, M. 2003. "What Is Geometry?" In *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching,* ed. C. Pritchard. Cambridge: Cambridge University Press, 29.

be unpolished and underdeveloped. Furthermore, though this hasn't been a hard and fast part of this criterion, we hope the evidence we gather of student geometric thinking across this grade band will reveal apparent developmental trajectories.

4. *The GHOMs should lend themselves to instructional use.* Our core interest is in helping teachers foster geometric thinking in and among their students. So our GHOM framework needs to be compact enough to be feasible and economical as a classroom resource. Further, each GHOM must point the way toward helpful instructional strategies—for example, productive questions to ask students and clues toward problem design and adaptation.

What are the *Fostering Geometric Thinking* GHOMs?

Working with the criteria listed above, we have settled on four GHOMs for these materials to emphasize.

1. ***Reasoning with relationships:*** Actively looking for relationships (e.g., congruence and similarity) within and between geometric figures. Relationships can be between separate figures, whole figures and their parts, or concepts (e.g., area and perimeter). Internal questions include: "How are these figures alike?" "In how many ways are they alike?" "How are these figures different?" "What would I have to do to this object to make it like that object?"

> **Example:** Over the course of time, from elementary grades through middle grades, a student's thinking will expand in answer to the question, "Which two make the best pair?" (See Figure 1.) In particular, it is the reasoning about and with relationships that expands. Younger students may identify the two smaller rectangles as those making the best pair because they are the two closest in size. However, as students develop an attention to mathematical relationships, they will begin to notice things like the proportional relationship between the side lengths of medium and large rectangles.

To foster students' habits of mind, teachers need to be alert for indicators of geometric thinking that show potential in problem solving. For *Reasoning with relationships*, we have noted the following indicators, some of which are focused on relationships between separate figures and others that are focused on subfigures within a single figure. Finally, there are special reasoning skills, particularly using proportion and symmetry. Students are *Reasoning with relationships* when they:

Focus on relationships among separate figures, by . . .

- comparing two or more geometric figures by enumerating some properties they have in common (which may or may not

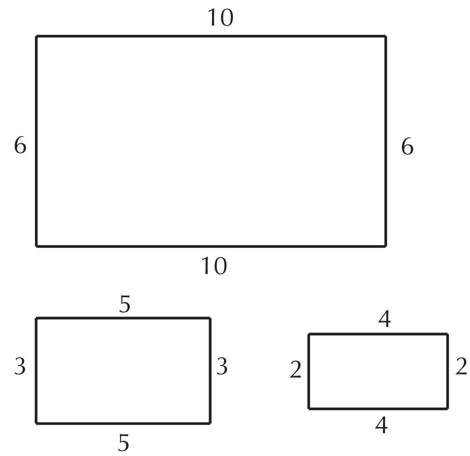


Figure 1

be relevant to the problem; e.g., relating two right triangles by the Pythagorean relationship, $a^2 + b^2 = c^2$)

- comparing two or more geometric figures by enumerating all properties they have in common (relevant to the problem) and why (e.g., using the equivalence of corresponding sides and angles in congruent triangles)
- contrasting two or more geometric figures by noting properties they do not have in common (e.g., recognizing that the Pythagorean relationship among triangle sides is unique to right triangles)
- comparing two or more geometric figures by considering relationships for their one-dimensional, two-dimensional, or three-dimensional components (e.g., relating the side lengths of similar triangles, as well as the areas)

Focus on relationships among the pieces in a single figure, by . . .

- noticing and relating subfigures within a geometric figure (e.g., looking at a geometric puzzle and seeing that a subset of pieces form a rectangle)
- constructing subfigures within a geometric figure (e.g., connecting vertices in a polygon to divide it into a set of triangles)
- relating two geometric figures by noticing they can be seen as parts of a single geometric figure (e.g., "If I extend these two line segments, they will become two of the sides of a rectangle"; "If I put these two pieces together, they form a square.")

Use special reasoning skills to focus on relationships by . . .

- reasoning proportionally about two or more geometric figures (e.g., "One of these triangles has sides that are 1.5 times as long as the sides of the other triangle. From that, I can figure out the relationship between the areas of the two.")

- using symmetry to relate geometric figures (e.g., "The altitude of this isosceles triangle divides the triangle into two triangles, one the mirror image of the other.")

2. ***Generalizing geometric ideas:*** Wanting to understand and describe the *always* and the *every* related to geometric concepts and procedures. Generalizing progresses through the following stages: conjecturing about the *every* and *always* and *how many cases,* testing the conjecture, drawing a conclusion about the conjecture, and making a convincing argument to support the conclusion. Internal questions include: "Does this happen in every case?" "Why would this happen in every case?" "Have I found all the ones that fit this description? [emphasis on *all the ones*]" "Can I think of examples when this is not true, and, if so, should I then revise my generalization?" "Would this apply in other dimensions?"

> **Example**: After drawing diagonals in squares of varying sizes, a student may look across all of her examples and notice, "In squares, the diagonals always intersect in 90-degree angles." This kind of insight into generalization can be powerful in geometric problem solving. (See Figure 2.)

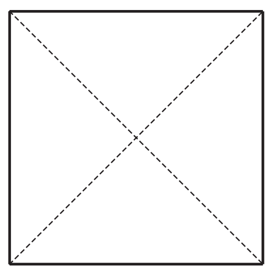


Figure 2

Overall, as we collected work on *Toolkit* problems and located indicators of *Generalizing geometric ideas,* we noted several levels in how far solvers go in determining whether they "have found them all" or whether a procedure or result "always works." Students are *Generalizing geometric ideas* when they:

Seek solutions from familiar cases or known solutions, by . . .

- considering relevant special cases (e.g., right triangles, equilateral triangles, whole-number side lengths)
- looking beyond special cases to some other examples that fit (e.g., trying a side length that is not a whole number)
- generating new cases by changing features in cases already identified (e.g., applying reflections, rotations)

- intuiting that there are other solutions, without knowing how to generate them (e.g., "There must be other points that work but their coordinates won't be nice numbers.")

Seek a range of solutions using assumed simplifying conditions, by . . .

- recognizing that the given conditions work for an infinite set, but considering only a discrete set (e.g., using points on a graph that have only integer coordinates)
- seeing an infinite, continuously varying set of cases that work, but limiting the set (e.g., by looking only within a bounded space in the plane) or jumping to the wrong conclusion about the set (e.g., by representing the set with the wrong geometric shape)

Seek complete solution sets or general rules, by . . .

- seeing the entire set of solutions and explaining why there are no more
- noticing a rule that is universally true for a class of geometric figures (e.g., "If you double the size of all the sides of any polygon, you quadruple the area.")
- situating problems or rules in broader contexts (e.g., "I bet a similar thing happens in three dimensions—if you double the edges of a polyhedron, you make the volume go up eight times.")

3. ***Investigating invariants:*** Analyzing which properties of a geometric figure are affected by a transformation (e.g., reflection, rotation, dissection). *Invariants* remain unchanged, even as other things vary. Properties of a figure that might be invariants during a transformation include a figure's orientation, location, area, perimeter, volume, side lengths, ratio of side lengths, and angles. Internal questions include: "How did that get from here to there?" "What changes? Why?" "What stays the same? Why?"

> **Example:** As an extension of the previous example, someone might do a thought experiment and imagine the square collapsing into flatter and flatter rhombi, wondering what changes and what stays the same. Area changes as the shape varies but perimeter does not. And neither does the angle of intersection between the diagonals! (See Figure 3.)

Overall, as we have looked for indicators of this GHOM, *Investigating invariants*, we have noted cases where the thinking is about *searching* for invariants and cases where attention is given to *checking* the effects of carrying out transformations on figures. Students are *Investigating invariants* when they:

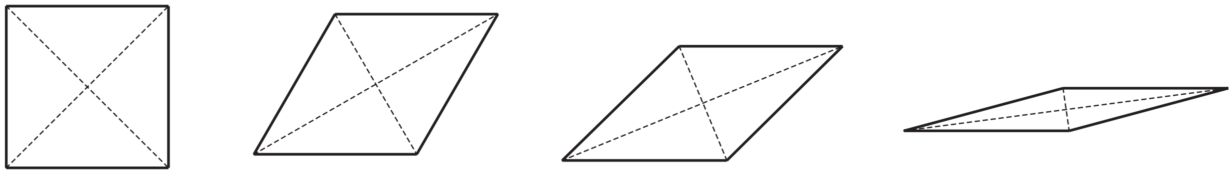


Figure 3

Use dynamic thinking and searching, by . . .

- thinking dynamically about a static case (e.g., "I wonder if it will be easier to figure out the area of this figure if I cut it up and move the pieces around.")
- wondering about what changes and what stays the same when a transformation is applied (e.g., "When I rotate a line segment around this point, what happens to the midpoint—it stays in the middle, right?")
- generating a number of cases of transformation effects and looking for commonalities (e.g., "We've dilated this triangle × 2, × 3, × (.5) and recorded what's changed and what hasn't.")
- thinking about the effects of moving a point or figure continuously and predicting occurrences in between one point and another (e.g., "Here's a triangle with perimeter 12 and area 6, and another triangle with perimeter 12 and area 4. There must be a triangle with perimeter 12 and area 5 somewhere in between.")
- considering limit cases and extreme cases under transformations (e.g., "What happens to the diagonals' intersection point as this figure collapses to a line segment?" "As the top vertex of this triangle moves around a circle, and the other two vertices stay put, I wonder at what point the triangle's area is largest.")

Check evidence of effects, by . . .

- intuiting that not everything is changing as a transformation is applied (e.g., "Each time we dilated one of these triangles, we got one that seemed to be like the one we started with—just bigger.")
- noticing that the same effect appears to happen each time a particular type of transformation is applied (e.g., "Each time we dilated one of these triangles, the angles seemed to stay the same.")
- noticing invariants when a transformation is applied and explaining why they are invariants (e.g., "When you reflect a triangle through a line, you get a triangle that's congruent.

That's because reflecting is like paper folding, and you don't change the size or shape of figures when you move them by folding paper.")

4. ***Balancing exploration and reflection:*** Trying various approaches (often chosen as a result of proposed hypotheses) and regularly stepping back to consider what has been learned. It's important that there is a balance between exploration, possibly guided by hypotheses, and reflection on what has been learned as a result of the exploration. Internal questions include: "What happens if I (draw a picture, add to/take apart this picture, work backward from the ending place, etc.)?" "What did that action tell me?" "How can my earlier attempts to solve the problem inform my approach now?"

    Example: Suppose you were given the following challenge: "Sketch, if it is possible, a quadrilateral with exactly two right angles and no parallel sides. If you think it impossible, say why." One way to think about it might go as follows: "I'll work backward and imagine the figure has been drawn. What can I say about it? One thing: the two right angles can't be right next to each other. Otherwise, you'd have two parallel sides. So, what if I draw two right angles and stick them together. . . ." This balance of "what if" with "what did I learn from trying that?" is representative of the fourth habit of mind—a balance of exploration with deduction.

Reflecting the double nature of this GHOM, the indicators we have noticed divide into two groups, depending on whether exploration or reflection about end goals is in the foreground. Students are *Balancing exploration and deduction* when they:

Put exploration in the foreground by . . .

- drawing, playing, and/or exploring through intuition or guessing (e.g., "This doesn't seem to work. Let's try something different.")
- drawing, playing, and/or exploring with regular stocktaking (e.g., "What did that tell me?")
- considering previous similar situations (e.g., "What have I tried before?")
- changing or considering changes to some feature of a situation, condition, or geometric figure (e.g., "What if I connected these two points instead of those two?")

Put end goals in the foreground by . . .

- periodically returning to the big picture as a touchstone of progress (e.g., "Now, how does that connect to what we're supposed to find?")

- identifying intermediate steps that can help get to the goal (e.g., "We know how to make a rectangle from a parallelogram, so if we can make a parallelogram out of this figure, we'll have it.")
- describing what the final state would look like (e.g., to see if there is any way to reason backward, such as "Well, I know the final set of points will be symmetric about the *y*-axis, so what might that shape look like?")
- making reasoned conjectures about solutions, creating ways to test the conjectures (e.g., "All the points that work will be symmetric about the *y*-axis. I think that means it will be a circle. To test that, we need to decide where the center of that circle would be, and then draw the circle and find out if points on it work.")

For an example showing how the four habits of mind can be helpful in conjunction with each other, consider the geometric situation in Figure 4. Student A connects the midpoints of two quadrilaterals and notices that the resulting figures in both of these examples seem to form a parallelogram. *Reasoning with relationships*, he then sets out to test if this relationship continues with other quadrilaterals (perhaps by using dynamic geometry software to distort the two examples, then by trying fresh examples). Each time, he notes that the apparent "parallelogram-ness" is held invariant. This line of thinking is consistent with *Investigating invariants*. Though it is a mystery to him, he starts to get curious as to whether this *always* happens (a sign of *Generalizing geometric ideas*).

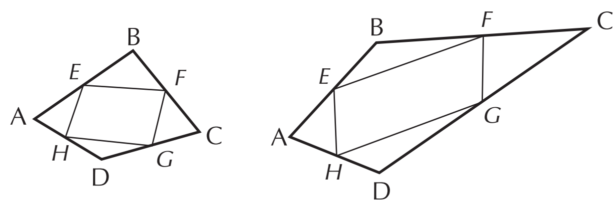


Figure 4

Student A tells Student B about this interesting observation. She does some exploration on a range of very different quadrilaterals, then reasons: "I've tried quadrilaterals of very different shapes, and the midpoints seem to keep forming parallelograms. I wonder why. I know something about midpoints of triangles, so if I can involve triangles. . . ." She then draws a line, as shown in Figure 5. By taking stock during the exploration process, Student B manifests one way that that habit of mind can influence thinking. She then is in a position to seek out and reason with relationships among ΔGDH, ΔEBF, ΔADC, and ΔABC.

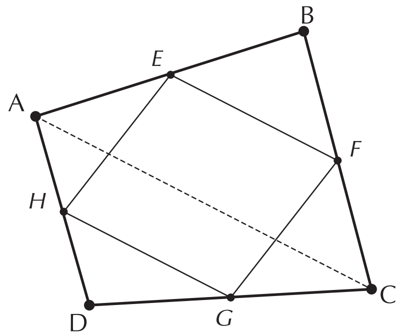


Figure 5

All four habits of mind do not necessarily apply to each geometric situation, nor do they apply in any fixed order. Any one of them can be helpful in jump-starting thinking on a geometric problem or, sometimes, on a problem that may not look "geometric," as the following example illustrates.

Several years ago, one of the Fostering Geometric Thinking (FGT) developers sat with a group of middle-grades teachers as they worked on the Staircase Problem[5]: "Staircase 1 has 1 block; Staircase 2 has 1 + 2 = 3 blocks; Staircase 3 has 1 + 2 + 3 = 6 blocks; Staircase 4 has 1 + 2 + 3 + 4 = 10 blocks; and so on. How many blocks are in Staircase *N*?"

Most teachers in the group focused on the sequence of numbers and compiled tables like the following:

***N*th staircase**	**# of blocks**
1	1
2	3
3	6
4	10
5	15
.	.
.	.

At this point, the teachers tried various moves aimed at revealing numerical patterns, like taking successive differences in the right-hand column, but generally they made no progress and were stuck. At the edge of the group, working by himself, one teacher was drawing pictures of each successive staircase. At a certain point, he let out an "Aha!," which he later reported as revealing his insight that each

[5]The Staircase Problem is drawn from the *Fostering Algebraic Thinking Toolkit* Analyzing Written Student Work module. It was originally developed by Al Cuoco and staff at Education Development Center, Inc., for use in teacher professional development.

staircase, while itself an irregular geometric figure, had a clear relationship with a very regular and familiar geometric figure, the square. Demonstrated for Staircase 3, that relationship is represented in Figure 6. Now he could see that the square was composed of a staircase and the previous staircase. From a FGT perspective, we would say that the teacher was making use of the *Reasoning with relationships* habit of mind and looking for relationships between geometric figures.

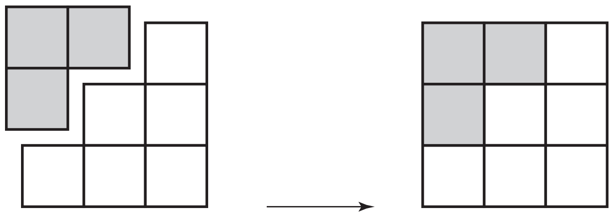


Figure 6

Throughout these materials, we will ask teachers to do something similar to what we did in describing these examples. We will ask them to track their own as well as others' thinking in this manner: What got their thinking started? What did they pay attention to? What did they recognize as they proceeded? and so on. In this way, we hope teachers will grow more accustomed to analyzing and describing their own geometric thinking, their colleagues' geometric thinking, and their students' geometric thinking.

Links to Classroom Content

Three Strands

The geometry addressed in middle school is conventionally organized and described around three topic areas, and the twenty sessions are organized into three groups, each focused on one of these three content strands.

- *properties of geometric objects* (i.e., objects like lines, triangles, etc.) with particular attention to how properties determine relationships among geometric objects—Sessions 2 through 7.
- *geometric transformations* and their effect on geometric objects, particularly invariance effects (e.g., Does a particular transformation preserve length of line segments? Does it preserve area?)—Sessions 8 through 13.
- *measurement of geometric objects*, considering measures like length, area, angle size, and volume—Sessions 14 through 19.

This organization into three content strands provides some topical links to classroom geometry instruction (e.g., the three content strands can connect to many areas of geometry, including symmetry, congruence, similarity, and rigid motions in the plane). We introduce a habits-of-mind lens on the content strands to identify lines of productive thinking in the strands—in particular, reasoning with relationships between geometric objects dictated by their properties; generalizing geometric ideas; investigating invariants; and balancing exploration and reflection.

We believe that this content/GHOM framework fits nicely with the Grades 6–8 Geometry and Measurement Standards of the National Council of Teachers of Mathematics (NCTM).[6] For example, key phrases used by NCTM in describing learning expectations related to those standards are:

- "analyze characteristics and properties of . . . geometric shapes and develop mathematical arguments about geometric relationships"
- "describe spatial relationships"
- "apply transformations"
- "use visualization, spatial reasoning, and geometric modeling to solve problems"
- "understand measurable attributes of objects"

Teachers will likely see other connections to their curricula and classroom practice. In particular, certain activities are built into these materials to help make connections to classroom practice (e.g., activities that focus on the role of language in developing productive GHOMs). In addition, in later sessions teachers will explore how to adapt mathematics problems to elicit more GHOMs.

Mathematics Problems for Classroom Use

An important part of the Structured Exploration Process is using the *Toolkit*'s mathematics problems with students, collecting student work, and then exploring the work with colleagues to look for evidence of GHOMs. Therefore, the particular mathematics problems provided in these materials play a critical role in the sessions. However, it is important to note that the eleven problems explored during the twenty sessions are not meant to substitute for your regular classroom materials. The *Toolkit* is a professional development curriculum, not a classroom curriculum.

The *Toolkit*'s geometry problems all share certain features: they are challenging, they generally include multiple possible entry points or extensions, and they help expose (and promote) students' geometric thinking. Workshop participants will also find that the problems vary in the GHOMs that are featured, the geometric topics that are explored, and the ways in which evidence of students' geometric

[6]National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 232, 240.

thinking will most likely be captured (e.g., through written work, videotapes, etc.).

At first glance, teachers may be concerned about being asked to use additional challenging problems with their students given the many demands on their time. They may wonder how the topics of the particular problems are relevant to the curriculum they are teaching. Our experience has shown that teachers using these problems with students can reap benefits for themselves and for their students in many forms, with carryover into many parts of the curriculum. The facilitator and teachers should attempt to maintain a stance of inquiry as they work with these materials. Viewing the use of the *Toolkit*'s mathematics problems in the classroom from this stance creates opportunities for learning about student thinking. In addition, teachers will have time to discuss connections between each problem and their own curriculum, as well as to consider adaptations they judge necessary in order to use the problems with their own students. Adaptations will be made with the aim of eliciting more of the students' current geometric thinking. Below are several concerns that teachers may have about using the *Toolkit*'s problems with their students, each followed by a response that grows out of the goals of the materials (see Goals section of the Orientation to Materials).

Teacher Concern: This problem is too difficult for my students. My students won't be able to solve this problem, or they won't be able to finish it in a single class period.

Response: The problems are challenging by design, so it is understandable for teachers to occasionally wonder if they are too difficult for their middle-grades students. Bear in mind that the goal of bringing these problems to the classroom is to help you gain insight into students' geometric thinking. Students often reveal promising ideas as they wrestle with challenging problems, even if they never get to a complete solution. In the context of these materials, students' developing ideas should be valued more than correct solutions as these ideas give teachers a foundation upon which to build further instruction. Also, keep in mind that students sometimes surprise us, so take advantage of this opportunity to challenge your students.

If you think that your students will not have any access to the problem, it is okay to make some adaptations. If adaptations are made, however, it should be done in such a way as to provide students an entryway into the challenges of the problem. Adaptations should never "water down" a problem or detract from its investigational nature.

Teacher Concern: My students have not learned this geometry topic yet!

Response: If your students have not yet been exposed to, or finished learning about, the particular geometry topics highlighted by the problem you can use the problem as a preassessment. This is a unique opportunity to explore the development of student thinking by learning about how students think about this problem before learning the topic. You can take the pressure off students by presenting it as a nongraded assignment—let students know that you want to find out about how they think through this problem because you are trying to learn about geometric thinking yourself.

Teacher Concern: We already covered this topic in September!

Response: In addition to the possibility of using *Toolkit* problems with students as preassessments, the problems can also be thought of as teaching tools when students are in the midst of learning about the particular geometry topics touched upon in the problem or as a check on current student understanding after students have already had exposure to the geometry topics. Also consider that most of the problems touch upon content areas besides geometry, so connections can be drawn between current work by students and the *Toolkit* problem.

Teacher Concern: This is a sixth-grade topic and I teach eighth grade—how is this relevant to me?

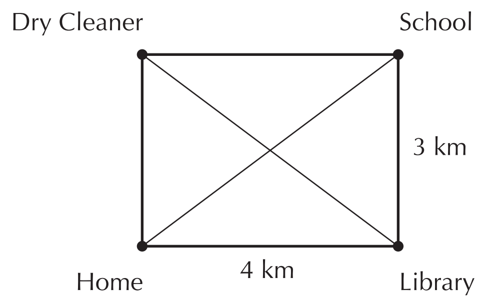
Response: First, consider the previous responses about using the *Toolkit* problems with an eye toward inquiry, in order to learn about the development of student thinking. For example, one way you and your colleagues can adapt a problem is to build in an extension that reaches to higher grades. In addition, think of the use of this problem as an opportunity to learn about the development of topics through the middle grades. You will be able to combine your own experiences and observations with those of colleagues teaching in other grades to better articulate pathways of student thinking through the middle-grades curriculum.

Teacher Concern: I'm not sure I fully understand this problem myself. How can I be expected to bring it to my classroom?

Response: Genuine mathematical problem solving is never a smooth ride from problem to solution. Students often fail to recognize this, however. They usually witness their teachers approaching mathematics in a very accurate and efficient way, and this often leaves the mistaken impression that if one can't instantaneously apply an appropriate approach to a problem, then the problem can't be solved. It can actually be quite beneficial for your students if you, as the teacher, are struggling with a problem. You can model a mature process of intelligently approaching the problem, learning from mistakes, persevering, communicating with others, and so on. You don't always have to be the expert!

Teacher Concern: I already have the added demand of state tests—how can I make time to pursue these extra problems with my students?

Response: While the *Toolkit* is not intended to be a test-prepatory program, we believe that the learning encountered by both teachers and students through this program will have the added benefit of enhancing student readiness for various forms of assessment related to geometry. It is widely accepted that students should be assessed on more than their ability to recall facts, and most state tests include items designed to elicit higher order thinking. Such items often appear in an open-response format on standardized tests. The *Toolkit* problems are similar in nature to these items, and hence provide students exposure to and practice with the process of explaining and justifying one's thinking. That said, many states rely solely on multiple choice items, but even these items can push beyond factual or procedural recall and evoke the types of thinking promoted by the *Toolkit*. For example, consider the multiple choice item shown below:



The figure above shows four locations in a town and the roads connecting them. The four outer roads form a rectangle, and the two inner roads cut across the rectangle diagonally. As shown in the diagam, the distance from Home to the Library along one of the outer roads is 4 km, and the distance from the Library to the School is 3 km along another outer road.

Mr. Brown is at home. He needs to pick up his daughter from school, and after picking her up he needs to stop by the library and the dry cleaner (in any order) before returning home. What is the shortest possible distance he can travel as he runs these errands?

(A) 14 km (B) 16 km (C) 18 km (D) 20 km

The "meat" of this particular problem, in terms of mathematical content, is the Pythagorean theorem. However, this is far from a straightforward "find the unknown side" type of problem. Success on this item demands that students interpret the problem correctly, recognize that the shortest distance from Home to School is along a diagonal, explore various possible pathways, recognize the applicability of the Pythagorean theorem, and so on. Such integrated thinking resonates with the habits of mind promoted in the *Toolkit*.

Introduction II

Facilitation Guidelines

The *Toolkit* materials are designed to provide you, as facilitator, with the necessary support to lead a group of teachers as they explore mathematics and student thinking, working toward the goals described earlier in the Orientation to the Materials. In the following Using the *Toolkit* Materials to Facilitate section, you will find a description of how the session materials will guide you as a facilitator. After this overview of the materials, the remaining sections of the Facilitation Guidelines detail the different roles you will find yourself in when you facilitate *Toolkit* sessions and give you examples of how those facilitation roles look in practice.

Using the *Toolkit* Materials to Facilitate

The *Toolkit* sequence consists of twenty sessions. There are two session types: Do Math sessions focus on engaging teachers in mathematical problem solving, and Analyze Student Work sessions entail investigations of student work. Accompanying several of the sessions, there are computer applets to aid the mathematical problem solving and/or video clips that demonstrate mathematical problem solving from the perspective of the student. These applets and video clips are on a DVD-ROM included with the materials.

All sessions begin with an Agenda, an Overview, instructions for Preparation, and a list of Materials to Gather for the session. The Agenda lists the given session's activities and timeline. Notice that only 110 minutes are scheduled in the Agenda. There are a number of ways to use the unscheduled ten minutes: (1) they can remain unscheduled, knowing that some activities may run over their allotted time, (2) you can designate them as break time for your group, or (3) you can schedule five minutes of the time as a break and leave the remaining five minutes unscheduled in anticipation of running over time on some session activities. The next section after the Agenda of each session is the Overview. The Overview presents the main mathematical concepts and GHOMs explored in the session, as well as the overall goals of the session. Finally, the Preparation section conveys practical advice on how to prepare for the session and the Materials to Gather section lists needed materials. After these four sections, you will find the activity instructions and notes that you will need to facilitate the session.

Each session is divided into three groups of activities. Within each activity grouping you will find the following information to support your work as a facilitator:

- *Materials for Activities:* Each group of activities begins with a list of Materials for Activities. This list contains the subset of materials from Materials to Gather that applies to these activities.
    - *Where do I find the handouts listed in Materials?* You can find black-line masters for any handouts on the DVD-ROM, arranged by session.
- *Activities:* For each activity in the group you will find several kinds of information.
    - *The time* you should spend on the activity.
    - *The work group size* for the activity (e.g., full group, small group, or individual).
    - *The purpose* of the activity.
    - *Instructions* for the activity. The instructions include directions about what to ask the teachers to do, specific questions to ask the teachers, and prompts to display.
    - *Notes* containing additional information that you may find helpful as you plan your facilitation of the activity.

      Support Notes contain background information and tips about facilitation.

      Technology Notes contain pointers to places where technology may enhance the exploration of a particular mathematics problem (for students and/or teachers). The use of technology is not required but the Technology Notes will help you think about places where it may be useful if you have access to it.

On the DVD-ROM, you will find master copies of all of the handouts needed for that session. The final handout in each Do Math session is a collection of math notes related to that session's problem. The Math Notes contain mathematical background for problems and links to the GHOMs. The Math Notes serve two purposes: (1) When you read them before a Do Math session, they provide you with extra background on the mathematics of the problem and help you with the process of facilitating your teachers' mathematical exploration and discussion. (2) When you distribute them to your teachers after a session is completed, the notes serve to describe the mathematics in more detail and/or demonstrate additional ways of exploring the problem.

In addition to the handouts, computer applets, student work samples, video clips, and research summary slideshows are also included on the DVD-ROM. The computer applets are associated with five of our geometry problems, four are intended to be utilized by teachers during the Do Math sessions associated with those problems, while one is simply referenced in the Math Notes for the given problem. The applets are also intended for students when teachers try the problems out with them.

The student work samples on the DVD-ROM include student written work for selected problems. Two samples of written work are included with each session relying on the analysis of written student work. The pieces of work range from typical to exceptional, and represent competence as well as conceptual difficulties. We have included exceptional pieces because student work that surprises you often points you back to aspects of the mathematics worth exploring further. We have also included some more typical pieces of work because there is value in looking for common signs of competence and difficulty, and understanding the thinking those students are doing. The primary purpose of the sample written work is to provide facilitators with a sense of how middle-grades students might approach the problems. Ideally you will select samples of written student work from your own site for the Analyze Student Work sessions, but you may also draw on the sample work provided in the DVD-ROM if you have difficulty obtaining useable student work.

Video clips of students collaborating on *Toolkit* problems are also available on the DVD-ROM. Some of the clips serve as background for the facilitator, demonstrating how some students think and explore as they work on the problem, while other clips are included as backup for those Analyze Student Work sessions where video is the student work to be analyzed. Those clips included as background videos are only meant to be viewed by the facilitator, as there is not sufficient time during the sessions for teachers to view them. Backup videos are provided for those Analyze Student Work sessions where video is the student work to be analyzed. Ideally, when video is the type of student work designated for analysis, you will be able to use video clips generated by one of your own teachers. However, in the event that you are unable to obtain an appropriate video from a teacher (e.g., if the video produced is inaudible or of poor quality, if it is exceedingly difficult to uncover student thinking, etc.), you may use the backup video for the Analyze Student Work activities. There is one additional type of video included on the DVD-ROM. In Session 5 there is a video associated with an activity focused on students' use of mathematical language. Note that for any transcripts presented with video clips in the materials, pseudonyms have been used for the students when possible (i.e., when students do not refer to one another by name during the video clip).

Finally, PowerPoint® slideshows summarizing existing research on students' geometric thinking on each of the three content strands are also available on the DVD-ROM. Written research summaries are included in the handouts for Sessions 2, 8, and 14 and the slideshows review the material in those summaries. While no formal time is built into session discussions of the summaries, if you have teachers who have not read the research summaries or who would benefit from an overview of the main points, you may wish to use these slides as a resource.

Leading Discussions of Mathematics

Underlying these materials is a belief that giving teachers regular opportunities to explore and solve mathematics problems, then to discuss together their different solution approaches, is valuable for teacher and student learning. Typically, teacher explorations are conducted in small groups, and the subsequent discussions in the full group. Productive explorations and discussions have the following salient features:

- At the end of the discussion, all participants have the sense of making progress, if not achieving a full understanding of a problem solution.
- Often, a hallmark of a productive exploration and discussion is the desire by participants to continue thinking about the problem after the session is over. This is particularly true when time expires before all participants reach a solution or understand particular solutions presented during the discussion.
- Participants describe their thinking to each other in small groups as well as in the full-group discussion. Clarity is valued and actively pursued, and explanations consist of a mathematical argument, not simply a procedural description or summary.
- Multiple ways of solving a particular problem are valued, particularly in the full-group discussion. Participants also value mathematical thinking that involves understanding relations among multiple strategies.
- During explorations and discussions it is safe to make mistakes. Beyond the element of safety, errors are valued because they provide opportunities to reconceptualize a problem, explore contradictions in solutions, or pursue alternative strategies.
- Collaboration on solving problems is valued and pursued. During explorations, collaborative work involves individuals holding themselves accountable for making their own thinking clear and

rigorous and for understanding all others in their group. During discussion, collaborative work also involves using mathematical argumentation to reach consensus about what constitute reasonable solutions.

- In the context of group discussions, opportunities to explore the GHOMs are seized frequently.

Eleven of the twenty sessions (the even-numbered sessions, plus Session 1) involve activities where teachers are exploring then discussing mathematics problems. Below are some concerns you may be feeling about facilitating these explorations and discussions of mathematical thinking, along with notes to help you think through those concerns.

Facilitator Concern: What role do I play in the exploration and discussion?

Response: In our experience, different facilitators choose to play slightly different roles during explorations and discussions of the mathematics problems. We recommend that you explore the mathematics problem on your own ahead of time and anticipate different approaches or strategies—this will allow you to pay more attention to what the teachers are thinking about as they explore the problem, and thereby make it easier to lead the discussion of the mathematics afterward. However, some facilitators choose not to explore the problem ahead of time in order to experience the process at the same time as the teachers. They feel this helps avoid the problem of teachers viewing them as "the expert" on the mathematics of the problem.

Facilitator Concern: This geometric habits of mind notion is new to me, too. I am concerned about expectations that I am an expert in it.

Response: Keep in mind that it is always okay to say, "I don't know." Teachers should know that you are developing your understanding of the GHOMs along with them. Your goal should be to maintain a focus on ways of thinking about geometry. In addition, know that you have a resource available to you—the Math Notes for each problem provide examples of common ways of thinking about the problems as well as some guidance about using the GHOMs framework to describe these ways of thinking.

Facilitator Concern: What if people make mathematical mistakes?

Response: First of all, wait to see if other participants pick up on the errors. Second, consider what kind of error it is. If it is a careless error, then telling the person is fine. If it is an error in interpretation, strategizing, and so on, then aim to have the person articulate the thinking that led to the mistaken direction. Generally, the mind-set that needs to be developed in the group is: We can learn from mistakes.

Errors create opportunities for inquiry—for example, "Why was it inviting to go in that direction of thinking? Was it something in the wording of the task, or in connections you made with previous tasks? Would students be prone to similar thinking?" Questions can help people reflect on what they've written, but questions should not be used only to call attention to mistakes.

Facilitator Concern: What if there are group members who do not, or think they do not, know the geometry content?

Response: Related to this facilitator concern are, most often, teachers' concerns that they are not adequately prepared to participate in a *Toolkit* group. A few thoughts come to mind from our experience. First, when teachers reveal this concern, they often believe their role in using the materials is to "teach" the mathematics. That is not their role, as we described in the section Mathematics Problems for Classroom Use. Sometimes, teachers believe that working on *Toolkit* problems requires considerable background knowledge, gained in formal geometry courses. That, too, is not the case. The problems are meant to elicit thinking, not to test how much content teachers have learned and retained. Indeed, one of our primary goals, you will recall, is that teachers will strengthen their understanding of geometry. You may find it helpful to reiterate this goal to your teachers.

Facilitator Concern: I'm a bit rusty on some of the geometry content myself. I feel nervous about leading teachers in a professional development seminar focused on geometry.

Response: It is not uncommon for facilitators to feel a bit apprehensive about their geometric content knowledge as they embark on a year of leadership with these materials. These fears are consistently allayed after the first few sessions, however. The Math Notes included at the end of each Do Math session are a primary and useful resource for refreshing your understanding of the material. Many facilitators have drawn support from their participating teachers, as well. There is typically at least one mathematically inclined individual in any gathering of teachers. These "resident experts" can be an invaluable resource to you and the group as a whole in terms of clarifying mathematical concepts.

Facilitator Concern: What if I don't know the curricula used by some of the teachers in my group?

Response: These materials do not assume this knowledge on the part of facilitators. Activities focused on the teachers' particular classrooms and curricula provide guidance on what to look for, collect, and so on.

Facilitator Concern: Time is tight in sessions, and the mathematics problems are challenging. I am concerned about teachers' frustration whenever they don't reach closure.

Response: Our belief is that there are different ways to reach closure, besides everyone arriving at "the answer" in the allotted time. For example, before transitioning to the next activity, reach group consensus on what is known about the problem solution and what is left to find out. Invite the group to continue working on it between sessions, and promise to set aside time during the following session to let those who make progress report out. Another route to closure: at the end of the session, you can distribute copies of the Math Notes for the particular problem to group participants.

Advising Teachers on Collecting Student Work

An important part of the *Toolkit* process is taking the mathematics problems to students and collecting student work that the group can analyze together. Although it is helpful if all teachers try the problems with their students, as it gives them firsthand experience with how middle-grades students think about the concepts in the problem, generally you will only be asking a few teachers to bring work back to the group. You will then select a few pieces from the work that has been submitted, and those pieces will be the focus of your Analyze Student Work sessions.

There are three types of student work that you and your teachers will be working with throughout the sessions: written work, five- to ten-minute video clips, and two- to three-minute video clips with accompanying transcripts. The materials specify particular types of work for the earlier sessions and then allow groups to choose the type of work they'll collect for later sessions. Descriptions of the three different types of student work, and tips for collecting quality work of each type, are listed next.

Written Work

Written work refers to writing in the form of text, symbols, or drawings that students produce while exploring a problem. Students may write on the problem itself, other paper they use while working on the problem (e.g., graph paper, patty paper, origami paper), as well as on newsprint or transparencies they use to share solutions with the rest of the class.

Tips for collecting written work

- Have students work in small groups.
- Ask students *how* and *why* questions as they work.
- Push students to document *how* they solved problems, not just their final solutions.

- Collect all written materials (e.g., graph paper, patty paper, origami paper).

Five- to Ten-Minute Video Clip

The video clip should capture the evolution of students' thinking over time with regard to their (1) use of one or more of the GHOMs and/or (2) understanding of a particular mathematical concept.

Tips for collecting video clips

- Determine if the camera's microphone is sufficient to capture students' discussions. You may need to use an external table microphone.
- Have three or four students work together in a group.
- Follow only one or two groups throughout the taping.
- Tape in a quiet space like a library, conference room, or empty classroom.
- Ask another student to be the videographer if finding classroom coverage is a problem.
- Let students know that they should talk with each other as they work on the problem (e.g., explain their thinking to each other, challenge each other if they disagree, ask questions of each other, etc.).
- Focus the camera on materials students are using when it's relevant to their discussion.
- Provide students with markers or colored pencils instead of lead pencils (because they show up better on video).
- Collect all written materials (e.g., graph paper, patty paper, origami paper, etc.).

Two- to Three-Minute Video Clip with Accompanying Transcript

The video clip should show a brief exchange between students that is particularly interesting because of what it reveals about students' (1) use of one or more of the GHOMs, (2) understanding of a particular mathematical concept, and/or (3) use of mathematical language. An accompanying transcript of twenty to twenty-five lines should be created to allow teachers the opportunity to examine students' thinking in depth after viewing the brief video. See previous tips for collecting video clips.

Tips for creating a transcript

- Track different students' contributions by preceding each statement with a pseudonym (e.g., Student 1 or Student A).
- If students refer to "this" or "that," provide a description in parentheses of what students are referencing.

- If gestures or actions are important to the exchange, include them in the transcript.

Leading Discussions of Student Work

The *Toolkit* materials have grown from the conviction that different kinds of evidence arising from classroom practice can shed light for teachers on how geometric thinking develops and how they can foster it in their students. The materials use several kinds of evidence, including individual written work; video footage of students working on problems; transcripts of what students say to each other while solving problems; and drawings and graphs produced by students during their work on problems.

It is important to distinguish different purposes for looking at evidence from student work, because different purposes lead to different kinds of discussions. For example, the purpose of understanding student thinking differs from the purpose of evaluating student achievement, which in turn differs from the purpose of improving instruction. All are important, and each should have its turn in teachers' professional development. Discussions during sessions are, for the most part, intended to be focused on understanding student thinking, with the intention of letting this increased understanding eventually inform instruction. In other words, the materials encourage thinking about shifting instruction *based on a deeper understanding of student thinking*.

Productive analyses and discussions based on evidence of student thinking have the following salient features:

- Participants feel safe. Discussions are guided by ground rules that keep people focused on the evidence, not on the teachers involved or particular students.
- The primary focus—in both analysis and discussion—is on student thinking and learning, not on evaluating student achievement, nor on teaching. (Some comments, questions, and suggestions about teaching strategies are bound to arise. They just need to play a secondary role.)
- Alternative interpretations of student evidence are valued and encouraged.
- Discussions present challenges to each person's beliefs, assumptions, and mind-sets.
- Discussions tease out important mathematical ideas underlying the evidence of student thinking.
- Participants base their interpretations of student thinking on the evidence provided.
- Based on their interpretations of evidence, participants make explicit connections to classroom practice.

Nine of the sessions (odd-numbered sessions except for Session 1) involve activities where teachers analyze and discuss student work. Following are some concerns you may be feeling about facilitating productive analyses of student work and discussions of student thinking, along with notes to help you think through those concerns.

Facilitator Concern: I'm concerned about the possibility for hurt feelings or defensive reactions when people bring in work from their own classrooms.

Response: If explorations of student work are conducted with a focus on the student's thinking and not on the teacher or student themselves, then this problem can usually be avoided. It is wise to be open with the group about this intention, consistently reminding the group to keep the focus on the student thinking represented by the student work. If a teacher does seem to be showing signs of defensiveness when listening to colleagues' interpretations of his or her students' work, make sure that teacher has the opportunity to share his or her own interpretations of the student work, but always with the focus on backing up any claims with evidence that can be seen in the student work. When teachers' comments are centered primarily on deficits they see in the student thinking represented in the work, you can re-steer the discussion toward the potential for growth in mathematical thinking that is evident for each student, which may reduce the likelihood for hurt feelings by keeping a more positive and forward-looking tone to the conversation. Deficits are important to note and discuss but so are signs of competence and potential. In any event, it is important to express your appreciation to the teachers for sharing the work from their classrooms, which allows the whole group to learn about students' mathematical thinking.

Facilitator Concern: If I'm getting the student work from the teachers at the same time that everyone else does, how can I make sure I lead a good discussion?

Response: Being faced with unfamiliar student work for the first time while teachers are exploring it can be challenging, particularly if different groups of teachers are looking at different samples of work. For this reason, we recommend that you establish a norm with the teachers in the group that they give you copies of the student work they will bring to the session a couple of days before the session, if at all possible. In this way, you will have time to review the student work on your own, before the session begins.

However, we realize that receiving the student work in advance will not always be possible. In these cases, it is helpful to take the quiet time when teachers are looking at the work to do the same yourself. We have found it helpful at such times to be asking ourselves questions like "What catches my attention?" "What do I find curious or confusing?" "Where do I see signs of GHOMs?" In a pinch, then, those can lead to discussion questions, like: "This bit caught my attention. What do you make of it? How would you characterize the student's thinking there?"

Facilitator Concern: I'm mainly used to scoring student work. How will I know if they are doing what they are supposed to be doing?

Response: Our instructions and support notes should provide some guidance. Listen to the tenor of the teacher conversations. Are they addressing issues other than the geometric thinking represented, such as issues of instruction, or whether the student "got it" or not, or whether the problem was fair or not? Such side issues arise naturally and require some attention. However, look for openings when you can ask a question that returns attention to trying to understand the geometric thinking that is evident in the student work. Possible questions that will be helpful in returning attention to student thinking are found in the facilitator notes for some of the activities, and in general, asking teachers to consider (or summarize) what they can say about the students' use of geometric habits of mind and how they can support those claims will be productive in keeping the focus on geometric thinking.

Facilitator Concern: What if people disagree about an interpretation?

Response: A mind-set behind these materials is that disagreement is good, if grounded in the evidence. Disagreement grounded in the evidence can elicit different beliefs and values held by teachers. A guiding premise that grows out of this mind-set is that it is often in raising beliefs and values to the surface, then reflecting on them in light of evidence, that teachers can sharpen and revise their instructional strategies. So, if members of your group disagree about the meaning of something they see in a piece of written work or in a video, ask each participant to restate their interpretation of what the student or students in question are thinking, and then to explain the particular evidence in the written work or video that supports their claim. Encourage the teachers to try to understand the alternative interpretations their colleagues are offering, and how those interpretations are tied to the evidence that is on the paper or in the words and actions audible or visible in the video. Emphasize that the goal is not to agree upon the "correct" interpretation, but rather to consider what can be learned about the mathematical thinking that the student or students may have been doing.

Facilitator Concern: What if *I* disagree with someone's interpretation?

Response: This is a real challenge because facilitators don't want to set themselves up as the "experts" in their groups. However, it is an opportunity to model the kind of evidence-based interpretations we want teachers to be making. So, one way we have handled the challenge is to say something like: "I thought of a different interpretation . . . and here is the evidence that led me to make it. . . ." This is often a valuable way to facilitate even when you do not necessarily disagree: "Another interpretation someone could make is . . . because of this evidence. . . ." In any case, it is important to point out that because the students are not actually present to reveal their thinking, these are *interpretations* and not the truth.

Managing Group Processes and Group Dynamics

Behind the *Toolkit* materials is a belief that teachers' learning is enhanced by working with colleagues who question, challenge, support, and provide a network of resources to one another. In our experience, teacher learning communities of this type have several salient features. These include:

- a sense of group direction or focus for work, developed and sustained by a capacity to be reflective about practice
- a desire to monitor changes in the group's knowledge, beliefs, and attitudes
- group norms that encourage questioning, challenging, and supportive interactions
- a genuine collaboration among all members, including the group leader
- an outward and an inward focus, reaching out beyond the group for ideas and energy that help drive continuous group learning

Issues of group process or group dynamics can arise in any session. It is particularly important to pay attention to these issues in early sessions, when norms for working together are being established, but you should keep an eye on the functioning of the group throughout the year. Following are some concerns you may be feeling about managing group process and dynamics, along with notes to help you think through those concerns.

Facilitator Concern: Many of the members of my group are friends with each other and/or my friends—how will I keep us focused on our work together without getting distracted?

Response: Keeping discussions from getting sidetracked into outside issues (whether they be personal ones arising out of friendship or other issues related to school) can sometimes feel difficult. Naturally, when teachers are given time to talk with each other, they can think of lots of things they want to talk about. However, in our experience it is helpful to rely on the structures of the professional development to help keep your work focused. For example, point back to the agenda and express the concern that you will not be able to get through all of the activities of the session if you do not stay focused on the current activity. Also, refer back to the group norms that the group established together in Session 1 to discuss whether the group is maintaining those norms and whether any additions or adjustments need to be made. Finally, it is sometimes helpful to be explicit about roles. For example, you can assign a member of the group the job of timekeeper. You can also talk about having your "facilitator hat" on and off, and be explicit about balancing being a participant in the group with making sure the group is functioning in a productive manner.

Facilitator Concern: What if my teachers come to the sessions unprepared?

Response: An important part of the Structured Exploration Process that your teachers will be engaging with throughout the sessions is the collection of student work for analysis, and therefore it can pose a challenge if teachers do not come to sessions with student work that they promised to collect. Try to make sure that everyone is on the same page about who is collecting what student work before the odd-numbered sessions. You may even want to check in with teachers before the session to see if they have had any problems in collecting student work. A feeling of engagement with the work by participating teachers is important to the process. We have found that when teachers have had the chance to analyze some work together in a nonthreatening setting, they find it a very worthwhile experience and will want to bring in more work from their students. If teachers are having trouble because of time demands, you may want to consider a rotating schedule for collecting work so that not all teachers need bring work to every session.

Facilitator Concern: I'm worried about negativity from some members of my group—how do I counteract negativity and keep it from hindering the group?

Response: We believe that a prerequisite to dealing with negativity is a commitment on the part of the facilitator to understand what is behind the negativity. We have mentioned that we believe an inquiry stance on the part of facilitator and participants is necessary to use these materials well, and that applies to this concern. When confronted by explicit expressions of "negativity," facilitators need to be willing to listen to the person or persons expressing them. This does not mean that you have to argue them out of their positions, nor does it mean that you have to solve the problems. However, in our experience, listening often reveals that the negative expression is based on a misunderstanding (e.g., about the materials' expectations of teachers, about teacher roles, etc.). Those misunderstandings can be cleared up. On the other hand, some negative expressions are not so easily addressed. In that case, it usually is best to acknowledge the person's perspective, and then move on. At some later time, you can check back in with the person, to see if things have changed. Most of the time, have such a conversation with the person outside of the session activities. Sometimes, expressions of negativity are not openly communicated but strongly conveyed in unspoken ways. There, too, you can talk with the person outside the group. Once again, be committed to listen.

Facilitator Concern: A few of the teachers who will be joining my group tend to be very vocal and dominate conversations. How can I make sure that everyone is able to participate equally?

Response: It is great to be aware of potential participation patterns in your group in advance, and it will be something to watch for over the course of your work. Different people have different styles of participation, and by using different modes of discussion and sharing, you can help all teachers to participate. To help you with this process, the materials include a variety of modes. In particular, you will notice that teachers are often asked to reflect in writing and/or work on a problem individually before beginning discussion with their colleagues. This individual reflection time allows people who prefer to gather their thoughts some time to do so before discussion begins. In addition, some structures for sharing require the facilitator to make more decisions about the order of sharing, which can allow you to draw in more teachers. Other activity structures rely on volunteers making their comments. Finally, maintaining a group norm of monitoring one's own and each others' participation in the group, and reminding the teachers of that norm, can help to distribute the responsibility of paying attention to participation over the whole group.

Facilitator Concern: How will I know if my group is doing well?

Response: To us, "working well" in these sessions means group members learning, working, and developing together. We have tried to aid you in your ongoing assessment by including in the materials teacher feedback forms for each session as well as occasional activities in which participants are asked to reflect on and express any confusions they have (e.g., about the GHOMs) and any new learnings. To give you some perspective on group evolution, drawing from our own experience with these and other similar materials, we can make the following predictions.

1. One indicator of group progress is the degree to which participants shift from attending only to correct solutions toward attend-

ing to the thinking behind solutions. The GHOMs will seem foreign at the start to many, if not all, of the group participants, and at that point they may not be able to identify GHOM examples in either teacher or student mathematical work. This is natural. After all, the habits-of-mind concept is foreign to most people who encounter these materials.

2. As time proceeds, participants may start to "see" GHOM examples in many places, and some of those citations will be inappropriate. Gradually, those participants develop a sharper eye for examples. To aid that development, during several different sessions we have included an activity that asks participants to look back over several sessions and to cite examples for each GHOM, in writing, about which they have little doubt. In addition, the activity asks them to cite the reasons and to discuss those reasons with others. To aid the development further, when a participant asks if part of someone's solution exemplifies this GHOM or not, you can express your opinion about the example (if you have one), or you can inquire into the participant's thinking about the thinking behind the example.
3. Balanced participation in discussions, over time, is another indicator of good group functioning. As we advise in another part of this Introduction, you can address imbalanced participation in several ways—for example, by structuring the reporting back and/or by addressing the dilemma explicitly.
4. In our own work, we also look at how seriously group members take the responsibility to bring in examples of student thinking. If the many leave the job to the few, then that has to be addressed in order for the group to develop and learn together.

Facilitation—Examples from *Toolkit* Groups

Your role as facilitator is complex, but with the support of these materials and with attention to the four facilitative roles described in the preceding section (i.e., leading discussions of mathematics, advising teachers on collecting student work, leading discussions of student work, and managing group processes and group dynamics), you will have the opportunity to both create a productive learning environment for your teachers, and to learn and grow in your role as facilitator.

The section that follows will give you a picture of how these facilitative roles might look during a session, by telling the story of a Do Math session in one school and of an Analyze Student Work session in another school. The facilitators at these two schools faced challenges, encountered success, and made on-the-spot decisions, similar to those you will experience during your own sessions. Examples of the challenges and facilitator choices and strategies during these sessions will give you a sense of productive uses of the materials by a facilitator and hopefully will reassure you that there is often more than one right answer for how best to handle a given situation. The examples from these two sessions are categorized according to the four facilitator roles described in the preceding sections.

Example I: The Story of a Do Math Session

Sandy[7] is a workshop facilitator leading a group of seven teachers from the school where she is math department head. She has three years of experience conducting professional development. During Session 6, Sandy's teachers explored and discussed the Puzzling with Polygons problem and also completed a Connect to Practice activity that addressed both how the materials are affecting the teachers' practice and the use of geometry tasks to elicit geometric thinking. Some of the facilitative challenges that Sandy faced and strategies that she used during this Do Math session are described below.

Leading Discussions of Mathematics (Staying Focused on the Goals of the Work)

One challenge that Sandy faced as a facilitator in Session 6 was how to convey the goals of the session to the teachers in the group. The materials suggest sharing the session goals with the group when reviewing the Agenda at the beginning of each meeting. To give the goals more weight, Sandy chose to display them in writing so that group members would be able to refer back to them at any point in the session. Before the session she had written the goals, as well as any other discussion prompts that are suggested for each activity, on chart paper. She taped the pieces of chart paper to the wall, folded in half, and as she was ready to share any piece of chart paper (i.e., the goals, or one of the discussion questions) during the session, she unfolded it so the writing was revealed to the group. Sandy understood the importance of staying focused on the main goals of the session and that some teachers will benefit from seeing these goals in writing rather than just hearing them once at the beginning of the session.

Although Sandy's display of session goals was intended to make teachers aware of the "big picture" purposes of the session, her teachers' understanding of the overarching intentions of the session was mixed. Two teachers who were interviewed after Session 6 described their view of the purpose of Session 6, and of all Do Math sessions.

[7]Names used in this section of the Introduction are pseudonyms, though all the examples are based on observations of *Toolkit* sessions and interviews with participating facilitators and teachers. We are grateful to the field-test teachers and facilitators who allowed us to observe some of their sessions and who gave their time to talk to us after those sessions.

Teacher A replied, "To learn the next project to present to the children." Teacher B responded, "To get a better understanding of geometric thinking and then try to relate it to our students." Although teacher B's response is more reflective of the session goals that Sandy had shared with the group, teacher A's response is not atypical, especially during early *Toolkit* sessions. It may take some time for teachers to see the value in unpacking their own geometric thinking about problems, rather than seeing Do Math sessions as an opportunity to get a new problem for use in the classroom. Sandy's continued focus on the session and activity goals will help both teachers to develop their understanding about the value of reflecting on geometric thinking.

Sandy's focus on what is important in any given session is also evident in her response to an interview question after Session 6, when she described her recent change in thinking about how to make best use of the session materials as she prepares for sessions and during the sessions:

> [I go] through the facilitator notes several times, and then I do checklists to make sure that I'm ready to go. And I use [the printout of the materials] now, where before I thought I couldn't use those sheets in front of me. I'd think "Oh, [the teachers will] think I'm cheating or something." But now I feel comfortable using them. . . . After every session I'll . . . sit and go over what I thought about it. And I'll think "I can't believe I forgot to bring up that point." And I don't want to do that. And [so now] I'll even highlight [in my printout of the FGT session materials] things that I definitely want to key in on [with teachers during the session].

Leading Discussions of Mathematics (Interacting with Small Groups)

Another part of Sandy's role as facilitator of mathematics discussions is her interaction with the small groups as they explore the mathematics problem—in this case, Puzzling with Polygons. During Session 6, Sandy chose to let the teachers' animated discussions in small groups about the Puzzling with Polygons problem continue relatively uninterrupted. Most of her interactions with the small groups involved reminding them to think aloud, so that their thinking would be evident to their colleagues. As facilitator, she could also have chosen to ask questions that pushed on the teachers' thinking in these instances, but the strategy of staying back and letting the teachers' thinking progress without interruption is also valuable, especially when a facilitator is trying to get a sense of her teachers' thinking.

In her post-Session 6 interview, Sandy described the development of her thinking over time about how to interact with the teachers when they are working in small groups on a mathematics problem.

> I still get nervous with the discussions. I feel like they have such rich discussions within their groups that I feel like I'm butting in. When I come around. Like I don't want to stop, so sometimes I just stand and listen, because . . . what they're offering, and just . . . some of the groups that are there, and being able to share and not yell and scream at each other that that's not right or wrong or whatever. I feel like if I stopped them they're not going to get that same communication and quality of openness. And this was not at all the way it started out. I had to pull things. It was almost like I was asking them leading questions. I didn't want to be asking them—but I was trying to get them to understand what this was all about, and now they've just gone so much further than my expectations.

During a later Do Math session when teachers were working on the Finding Centers of Rotation problem, it was evident that Sandy's emphasis on thinking aloud (as seen in Session 6) was starting to pay off, in that teachers were actively discussing their ideas throughout their work on the problem. Furthermore, by this later Do Math session, Sandy's view and/or ease with her role as facilitator of the small-group discussions in Do Math sessions had progressed to the point that she was using her observations about what happened in the small group to guide her questions during the full-group discussion of the problem (i.e., she asked a group that had thought about rotations in a very different way to share their thinking, and she highlighted the differences and similarities between that method and the other groups' strategies).

Managing Group Processes and Group Dynamics

Facilitating any professional development session involves making certain decisions related to the individuals in the group and related to how they interact with one another. An example of how this role of managing group processes arose during Sandy's Session 6 when it was time to split into small groups for exploration of the Puzzling with Polygons problem. Sandy asked the group to split into two small groups by directing all of the teachers wearing red shirts to form one group, and the teachers wearing white shirts to form another group. Although this method of creating small groups could have looked random to an observer, when asked about problems that she predicted for the session and how she dealt with them, she explained:

> Groups . . . if I ask them to get into groups, they'll always flock to the same ones. I find that difficult to work with because I'll just always get the same results each time so I

> have chosen to come up with a way to produce a grade range in each group.

Sandy had used the shirt color method for splitting groups because she saw in advance that on this particular day it would produce groups with a variety of grade levels represented and in which teachers would interact with different colleagues.

Advising Teachers on Collecting Student Work (Preparing Teachers to Use the Problems)

One final example of the facilitator role as seen through Sandy's work with her group relates to preparing teachers to collect student work. Some teachers, especially near the beginning of their work with these materials, are hesitant about using some of the geometry tasks with their younger students. Sandy noted some of this type of hesitation with her teachers, but through her enthusiasm and expressions of confidence in the teachers and in the students, she conveyed the message that the teachers should go ahead and try the problems out with their students, just to see what might happen. One teacher, when interviewed after Session 6, reflected back on the success of being encouraged to go through the experience of using problems with students by saying:

> I am finding that once you let the kids go and don't restrict their thinking, they actually surprise you in some of the things they come up with. We did an activity two weeks ago and I'm thinking "Wow, this is difficult, I don't know how much they're going to get of this" and I had two of the five groups really come up with some great ideas and strategies, and I thought they really did better than I thought they were going to do. I guess it just goes to show you that you've got to teach it to them and then let them see what they can do with it. I think we've been restricting [them, by thinking] "Oh, fifth graders won't understand that. . . . I won't teach it" and I think that's where we restrict them. Let the ones who understand it, understand it, then let's teach everyone, and then some of them it will take them a couple years to mature and then they'll get it eventually.

Example II: The Story of an Analyze Student Work Session

Anna is another facilitator working with a group of four middle-grades mathematics teachers from her school. Anna is a mathematics teacher with no experience leading professional development—she volunteered for the role of facilitator because she and her colleagues were very interested in participating, but their mathematics department head was unable to take on the role of facilitator. During Session 7, her group analyzed and discussed written work produced by their students as they worked on the Puzzling with Polygons problem and also completed a Connect to Practice activity in which they examined the role of language for students working on this problem, and finally they consolidated the ideas they had been developing about geometric properties during the first seven sessions. Some of the facilitative challenges that Anna faced and strategies that she used during this Analyze Student Work session are described next.

Leading Discussions of Student Work

During an interview after Session 7, Anna described some of her wonderings about how to take on the role of facilitator of a student work discussion focused on the GHOMs, while also being peer and colleague to the other group members. She explained:

> I feel personally comfortable [with the GHOMs]. I'm not sure I'm facilitating in a way that's bringing out the best in them. That's where I have more questions. Like, I can see the GHOMs, and I'm wondering "Am I leading too much? Should I let them struggle more? Should I step in? Should I not step in?" I find that, for me, that's the hard part. Defining my role as the facilitator. The materials themselves speak well and carry themselves. It's just—where do I have trouble stepping in or not stepping in.

During Session 7, Anna acted as one of the participants during the student work analysis. With such a small group and with one teacher absent due to illness, every teacher, including Anna, needed to be involved in discussing the thinking they saw in the student work that had been collected for the session. However, Anna did take on a role as facilitator by offering examples of GHOMs that she saw in the student work to get the discussion started when the other teachers remained quiet. She also played a role in guiding the group's discussion toward issues of language. Language is the prescribed focus of the second Analyze Student Work discussion in the Session 7 materials, but Anna chose to bring this focus to all parts of the student work analysis because she perceived the importance of this issue to her teachers.

Advising Teachers on Collecting Student Work (Selecting Work for Analysis)

As has been mentioned, Anna's group is small, so when one or more teachers are absent, every person plays a critical role in getting student work to the table for discussion. In Anna's Session 7, she collected a few pieces of written work from all of the teachers in the group, including the teacher who was absent for the session, which enabled her to have just the right amount of student work—a large enough sample from which to choose, but not too large a sample that would be overwhelming (she has a small enough group of teachers that collecting work from all teachers does not result in too large a pile). She chose to put a few pieces of chart paper that had been generated by students on a side table so that the teachers could look at them during any extra time in the session, and she selected a few interesting regular-sized pieces of student work for analysis. The chart paper would not have been as easy to photocopy for everyone to look at (though with such a small group it might also have worked to all gather around the chart paper), and she was able to find interesting pieces among the piles of regular student work she received from teachers.

Anna chose interesting pieces for analysis by focusing on language issues that she knew had come up in her group's discussions during Session 5, the session when issues of language in geometry are introduced. The Session 7 materials encourage the facilitator to choose for analysis "pieces that you think will generate a lot of discussion about students' developing understanding of the mathematics ideas, students' use of the GHOMs, and students' developing understanding of convincing mathematical arguments." So, Anna's focus on language during the student work selection is slightly different from the proposed criteria, but her focus is very much in line with the intent of the second part of the student work analysis during Session 7, when teachers are asked to think carefully about student's use and misuse of language in the task. In addition, Anna took her focus on the language to a deeper level by zeroing in on pieces of student work that she knew highlighted language issues in which her teachers were particularly interested.

As you move on to facilitate your group through the Do Math and Analyze Student Work sessions, keep in mind your colleagues around the country, including Anna and Sandy, who are facing similar decisions and challenges, but who, like you, can always fall back on the support provided in the materials when in doubt about what to do. It will be an interesting learning process for your teachers, and for you, as you explore this world of geometric thinking.

Introduction III

Organizing Your Group

What You Need

- *A facilitator*. The *Toolkit* materials are written such that a facilitator guides a group through each session. The facilitator can be a teacher leader, administrator, or an experienced classroom teacher. The main criteria for selection of a facilitator include a familiarity with and curiosity about middle-grades geometry and some experience with leading study groups or professional development (or a willingness to learn about this type of facilitation). The facilitator need not be an expert in geometry, though a strong background in mathematics will serve the group well. In the field test of the materials, facilitators with a range of facilitation experience and mathematics expertise led the study groups. Depending on the knowledge and experience of the facilitator, he or she will rely more or less heavily on the supports provided in the materials, which include detailed notes and brief tips about the mathematics, and notes and tips related to monitoring group discussions, promoting the sharing of ideas, selecting interesting pieces of student work for analysis, and helpful information on many other aspects of facilitating a group. An additional resource for facilitators wishing to develop their capacity for leading mathematics professional development is *Learning to Lead Mathematics Professional Development* by Catherine Carroll and Judith Mumme.[8]

[8]Carroll, C., and Mumme, J. 2007. *Learning to Lead Mathematics Professional Development.* Thousand Oaks, CA: Corwin Press and WestEd.

- *A group of middle-grades teachers.* The materials are designed for groups of six to twenty middle-grades teachers (grades 5 through 10). Significantly smaller or larger groups pose additional challenges. Smaller groups may have difficulty maintaining rich discussions, and larger groups may find it difficult to manage participation. We believe materials can be used with larger groups if some adaptations are made (e.g., adding a second facilitator). Ideally, groups should include a mix of teachers from across the six middle grades, because having several grades represented provides an opportunity for varied perspectives on the math and student work components of the *Toolkit* materials. It also allows teachers to gain a greater understanding of the development of students' geometric thinking over the middle grades. A form letter describing the program to interested teachers is included later in this section for recruitment.
- *A meeting structure*. Your work can be structured as ten four-hour monthly meetings or twenty two-hour biweekly meetings. Those groups that choose the ten-meeting structure complete two sessions per meeting and those that choose the twenty-meeting structure complete one session per meeting, as there are twenty sessions in all. The ten-meeting structure may work best for groups with: (1) a half day each month designated for professional development, (2) at least one early release day per month, or (3) the ability to devote part of one Saturday a month to the sessions. In other cases, the twenty-meeting structure may prove more feasible. A combination of the two meeting structures is also possible, with a mix of two-hour and four-hour meetings that result in a total meeting time of forty hours. (The first session for a four-hour meeting must always be an odd-numbered session, where participants look at student work gathered after the previous session.)
- *A regular meeting time*. These materials are intended to be used by a group of teachers meeting regularly throughout the school year. A schedule of the year's meetings should be set at the beginning of the school year. Groups applying a ten-meeting structure should designate a four-hour period of time to meet each month, and groups applying a twenty-meeting structure should designate two two-hour periods of time to meet each month. Analyze Student Work sessions should be scheduled at least two weeks after Do Math sessions as teachers will need time to do geometry activities with their students. You may even consider allowing three weeks between these sessions (meaning that only one week would separate Analyze Student Work and Do Math sessions). You may also want to plan for the unexpected by discussing a strategy for

making up meetings or scheduling a couple of extra sessions. Use the Meeting Calendar at the end of this section to lay out your meeting times and locations. We recommend consulting with school administrators as you set your schedule. Administrative support for the professional development time is critical to ensure that your scheduled times are protected.

- *A meeting place.* Your meeting place should be large enough to accommodate your group comfortably. The *Toolkit*'s activities often involve working in small groups. Therefore, the meeting place should also be able to accommodate several groups of three or four teachers. Arranging the tables or desks so that people can face each other during both large-group and small-group activity can facilitate discussion. It would be ideal, but not imperative, for the room to contain a blackboard and/or overhead projector, as reflection and discussion prompts will need to be posted or displayed regularly during sessions. Finally, several sessions have the option of using computer technology, and for those sessions you might consider meeting in a room with computer access.
- *Administrative support.* Utilize the letter included later in this section to inform the principal, mathematics coordinators and appropriate administrators of your plan to engage with the *Toolkit* materials and to seek their support in protecting the scheduled meeting times for the group. You may also want to invite administrators to sit in on some sessions to learn about the process—it is probably best to extend this invitation after the group has been working together for at least a few sessions.
- *Materials for participants.* All participating teachers should have a binder to organize their workshop materials. Prior to each session, you will need to print out and make copies of several handouts for participants. Every participant should bring their binder to each session, so if you three-hole-punch the copies it will make it easier for participants to store the handouts for reference in later sessions. Participants will also sometimes need access to various mathematics resources and tools as they explore mathematics problems during the sessions. Each session contains a Materials to Gather section with a list of everything needed for that session, but for your reference, here is a summary of the materials needed throughout the twenty sessions:

Necessary

binders	blank paper
patty paper	two-and-a-half-inch elastic bands

string

straws

toothpicks (for optional problem)

marshmallows (for optional problem)

highlighters and markers

straightedges

scissors

sticky notes in four colors

equipment for showing video (e.g., computer with DVD player, TV and DVD player, TV and VCR)

Toolkit materials and photocopier

overhead projector and blank transparencies and/or chart paper[9]

Very Helpful[10]

tangram sets

origami paper

graph paper

compass

protractors

colored pencils

computer access

Geometer's Sketchpad

refreshments

Start-up Materials

Letter to Administrators

As you begin to organize a group of teachers to work with the *Toolkit* (and later, as you work with those teachers), it is important to stay in communication with administrators in your district and to maintain their support. Administrators can be a key resource for gaining and holding meeting time and meeting space and for generating and maintaining enthusiasm for the work. In this section, you will find a sample letter to administrators that you can use to introduce your administrators to the program.

Letter to Prospective Participants

When recruiting teachers to participate, potential participants will have questions about what involvement in the group will entail and

[9]We strongly recommend the use of overhead projectors and chart paper for sharing work in different sessions, but if you cannot obtain chart paper, you may adapt how teachers share work in those sessions.

[10]The very helpful items will greatly assist teachers as they work on certain activities, but they are not as critical as the items on the necessary list, so focus first on gathering the necessary items.

what they will gain from the process. This section includes a sample recruitment letter to use as you build your group.

Meeting Calendar

Once your group is finalized (or even as you're working to recruit teachers), you will need to set a schedule for the meetings and share it with participating teachers. It is a good idea to set the dates for all of the meetings at the beginning of the year, rather than waiting to schedule later meetings at a future date. We also recommend scheduling a greater portion of the meeting during the first half of the year. Then, if conflicts arise and a meeting or two has to be canceled, there will be time to reschedule during the second half of the year. In the pages that follow, you will find a sample schedule that you can use to list the dates and times of your sessions.

Each session is two hours long. Make sure that you allow enough time between each even-numbered (Do Math) session and the following odd-numbered (Analyze Student Work) session for teachers to gather student work from their classrooms. If you wish to combine two sessions into one four-hour meeting, remember that you must pair an odd-numbered session with the even-numbered session that follows it (you cannot pair sessions the other way around because teachers need time to gather student work in their classrooms between each even-numbered session and the subsequent odd-numbered session).

Note that the sample schedule has a column on the right-hand side entitled "Bring Student Work." Teachers will need to bring student work (on assigned *Toolkit* problems) to each session with a check mark (✓) in that column. If you decide to rotate who brings student work to these sessions (while making sure at least three people bring student work to each one), then use this column for notes about who will bring student work to which sessions.

Student Work Release Form and Video Release Form

You may want to have your teachers collect Student Work Release Forms from students with whom they anticipate using the *Toolkit* problems at the beginning of the year. (Most groups will probably not find it necessary to secure consent for written work, but a form is included just in case). A little later in the year, you will find session notes about distributing and collecting the Video Release Form that is also included here. Keep in mind that your district may have its own policies about videotaping students for teacher analysis.

Attendance Form

At each session, you should have all participants sign in, as you may need documentation of attendance at a later date.

Dear Administrator,

A number of your teachers have the opportunity to take part in Fostering Geometric Thinking (FGT), forty hours of professional development aimed at strengthening teachers' understanding of geometry and supporting teachers as they advance geometric thinking in their classrooms. FGT is designed as a series of group study guides offering the following:

- a conceptual framework to help teachers understand middle school students' thinking in geometry and measurement and to guide them in engaging students' thinking more productively
- hands-on investigation of rich mathematical problems in geometry and measurement and tools for discussion and reflection aimed at deepening teachers' understanding of geometric thinking
- structured approaches to gathering and analyzing data about how students' thinking about geometry and measurement develops
- structured approaches to discussion among teachers about mathematics, curriculum, student thinking, and other issues related to teachers' practice

The design of the FGT professional development materials is rooted in a model adapted for use in previously published materials such as *The Fostering Algebraic Thinking Toolkit*. The model uses challenging mathematics activities and artifacts of student thinking to prompt teachers to reflect on the nature of mathematical thinking from different perspectives. The model gives teachers opportunities to become more effective teachers of mathematics by paying attention to their own mathematical thinking, their students' mathematical thinking, and connections between their learning and their classroom practice.

FGT's forty hours of professional development materials are designed to be completed over ten four-hour monthly meetings or twenty two-hour biweekly meetings, depending on scheduling needs. As an administrator, you can serve as a resource for gaining and holding meeting times and meeting space and for generating and maintaining enthusiasm for the work. We hope that you will support your teachers in their efforts to further their own geometric knowledge and that of their students.

Sincerely,

Dear Middle-Grades Math Teacher,

As a middle-grades math teacher, you have the opportunity to take part in Fostering Geometric Thinking (FGT), forty hours of professional development aimed at strengthening your understanding of geometry and supporting you as you advance geometric thinking in your classroom. The design of the FGT professional development materials is rooted in a cyclical model adapted for use in previously published materials such as *The Fostering Algebraic Thinking Toolkit*. The model uses challenging mathematics activities and artifacts of student thinking to prompt teachers to reflect on the nature of mathematical thinking from different perspectives. The stages of this cyclical model are, in brief:

Stage 1: *Doing mathematics*. Teachers work together in a study group to explore and solve mathematics problems they will later use with their students.

Stage 2: *Reflecting on the mathematics*. Using an explicit conceptual framework (such as "habits of mind"), teachers discuss the mathematical ideas and their thinking about the problem.

Stage 3: *Teaching the mathematics*. Teachers use the problems in their own classes and collect artifacts (e.g., student work).

Stage 4: *Analyzing artifacts*. Teachers bring selected artifacts back to the study group to analyze and discuss with colleagues.

Stage 5: *Reflecting on students' thinking*. Once again using an explicit conceptual framework, teachers discuss students' mathematical thinking, as revealed in the artifacts, and ways to elicit more productive thinking in future classes.

The forty hours of FGT professional development materials are designed to be completed over ten four-hour monthly meetings or twenty two-hour biweekly meetings, depending on your group's scheduling needs. If you would like to be part of an FGT study group, contact ______________________________ at ______________________________.

Sincerely,

Meeting Calendar

Date	Time	Location	Session	Bring Student Work
			1. Introduction and Do Math	
			2. Do Math	
			3. Analyze Student Work	✓
			4. Do Math	
			5. Analyze Student Work	✓
			6. Do Math	
			7. Analyze Student Work	✓
			8. Do Math	
			9. Analyze Student Work	✓
			10. Do Math	
			11. Analyze Student Work	✓
			12. Do Math	
			13. Analyze Student Work	✓
			14. Do Math	
			15. Analyze Student Work	✓
			16. Do Math	
			17. Analyze Student Work	✓
			18. Do Math	
			19. Analyze Student Work	✓
			20. Conclusion and Do Math	

Fostering Geometric Thinking Student Work Release Form

Dear Parent or Guardian,

Your child's teacher has been selected to participate in Fostering Geometric Thinking, a professional development seminar that focuses on the improvement of students' geometric thinking. One of the key components to effective training and informational materials are examples of work from students. Teachers participating in this seminar will be provided with mathematics activities to use occasionally with their students in grades 5 through 10. We ask for permission to collect your child's work samples to help teachers understand students' geometric thinking and how to improve it.

You can indicate that you are giving permission for your child's work samples to be collected by completing the release form included with this letter. Two copies of this request are enclosed. Please sign both copies and return one to your child's teacher, keeping the other one for your files. If you do not wish for your child's work samples to be collected and convey that wish, their work will not be collected for use during the seminar.

If you have any questions, please contact ______________________________ at ______________________________. Thank you for your consideration.

Sincerely,

Please return this form to your child's teacher.

Parent/Guardian Release Authorization

____

Child's Name

____

Name of Teacher and School

The participants of the Fostering Geometric Thinking seminar request permission to collect your child's work samples. Your signature enables use of the work samples to help teachers understand students' geometric thinking and how to improve it. The work samples may be used for educational purposes only.

I confirm that I have carefully read this CONSENT AND RELEASE and agreed to its terms knowingly and voluntarily.

____________________ ____________________

Signature Date

____

Printed Name

Please keep this form for your records.

Parent/Guardian Release Authorization

____
Child's Name

____
Name of Teacher and School

The participants of the Fostering Geometric Thinking seminar request permission to collect your child's work samples. Your signature enables use of the work samples to help teachers understand students' geometric thinking and how to improve it. The work samples may be used for educational purposes only.

I confirm that I have carefully read this CONSENT AND RELEASE and agreed to its terms knowingly and voluntarily.

_______________________ _______________________
Signature Date

____
Printed Name

Fostering Geometric Thinking Video Release Form

Dear Parent or Guardian,

Your child's teacher has been selected to participate in Fostering Geometric Thinking, a professional development seminar that focuses on the improvement of students' geometric thinking. As part of this seminar, video footage of students exploring geometry problems will be collected to help teachers understand students' geometric thinking and how to improve it.

We'd like your permission for your child to be included in the videotaping. You can indicate this by completing the release form included with this letter. Two copies of this request are enclosed. Please sign both copies and return one to your child's teacher, keeping the other one for your files. If you do not wish your child to be videotaped and convey that wish, we will avoid taping your child during class sessions.

If you have any questions, please contact ______________________________ at ___________________________. Thank you for your consideration.

Sincerely,

Please return this form to your child's teacher.

Parent/Guardian Release Authorization

____

Child's Name

____

Name of Teacher and School

The participants of the Fostering Geometric Thinking seminar request permission to videotape your child at school. Your signature enables use of the videos to help teachers understand students' geometric thinking and how to improve it. The videos may be used for educational purposes only.

I confirm that I have carefully read this CONSENT AND RELEASE and agreed to its terms knowingly and voluntarily.

______________________	______________________
Signature	Date

____

Printed Name

Please keep this form for your records.

Parent/Guardian Release Authorization

____
Child's Name

____
Name of Teacher and School

The participants of the Fostering Geometric Thinking seminar request permission to videotape your child at school. Your signature enables use of the videos to help teachers understand students' geometric thinking and how to improve it. The videos may be used for educational purposes only.

I confirm that I have carefully read this CONSENT AND RELEASE and agreed to its terms knowingly and voluntarily.

_______________________ _______________________
Signature Date

____
Printed Name

Fostering Geometric Thinking Attendance Form

School Name: ______________________________

Meeting Date: ______________________________

Meeting Time: ______________________________

Print Name:	Signature: