

Algebra and the Elementary Classroom

Transforming Thinking, Transforming Practice

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Foreword by Megan Loef Franke

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Changing How I Teach

When I look back at the way I had been teaching math, I know now that I was not a great math teacher. I remember standing in front of the class teaching math, and it was so unfulfilling I would often not want to teach it. If I skipped math some days, I didn't care. I was not making my students think. I was making them memorize strategies to subtract, add, and multiply. I wanted them to know their facts fast and automatically.

I would often stand in front of my students and teach them what was important for them to learn. There were many discussions in my classroom, but I did not allow the students to run or drive the discussion. It was my job to do that. I often allowed my students to run discussions when it came to literacy or science, but with math I just never thought of it.

I have now changed how I run my classroom. It is no longer about me, but about students. My math lessons are now student driven. My room also looks different and the writing that goes on my board when doing a lesson also looks very different. I now even have a board dedicated to math conjectures.

—Angela Gardiner, third-grade teacher

As with Angela, building a classroom where children think algebraically might change some (or many!) of the teaching practices you currently use. In fact, an important aspect of early algebra is that tasks and curriculum alone are not enough—your expertise as a teacher is essential in the development of children's algebraic thinking. This section looks at practices that

support children's thinking, including questioning strategies you might use, ways of listening to children's ideas to find and exploit algebra opportunities, ways to help children build and connect multiple representations of their thinking, and ways to help children generalize their mathematical ideas by forming and testing mathematical conjectures. As you read this section, compare your current practice with the ideas contained here. Look at the similarities and the differences, and develop a plan for how you might make changes.

Teaching Practices That Develop Children's Algebraic Thinking Skills

Algebraic thinking tasks alone will not give students the skills they need to reason algebraically. *How these tasks are used in instruction is equally important.*

In the discussion on the meanings children give the symbol $=$ (Chapter 2), we saw that tasks by themselves can lead to unintended learning that can even hinder students' algebraic thinking. On the other hand, a simple task like $9 + 3 = \square + 4$ can be a source of thoughtful classroom investigation if the teacher uses it to challenge students' understanding of equality. But what would this instruction look like? What would the teacher's role be? What would students' roles be? How can a teacher give a task more algebraic meaning for children? This chapter tries to unearth some of these skills and discuss specific ways for teachers to develop algebraic thinking in the classroom.

Represent, Question, Listen, Generalize

Classroom instruction that supports children's algebraic thinking is marked by rich conversation in which children make and explore mathematical conjectures, build arguments to establish or refute these conjectures, and treat established conjectures (generalizations) as important pieces of shared classroom knowledge. Moreover, this type of instruction treats these processes as a *regular* part of classroom activity, not an occasional enrichment to routines such as practicing arithmetic skills and procedures. In other words, algebraic thinking is a habit of mind that students acquire

through instruction that builds regular, sustained opportunities to think about, describe, and justify general relationships in arithmetic, geometry, and so on. There are four important instructional goals for you to keep in mind as you help children think algebraically:

- **Represent:** Provide multiple ways for children to systematically represent algebraic situations.
- **Question:** Ask questions that encourage children to think algebraically.
- **Listen:** Listen to and build on children's thinking.
- **Generalize:** Help children develop and justify their own generalizations.

Let's turn our attention to what these goals entail and what they might look like in instruction.

Represent: Provide Multiple Ways to Represent Algebraic Situations

As I walk around to each group, students are doing something different to solve the problem. Some groups are drawing pictures, some are using words and symbols, while others still are working on charts and graphs. It is quite refreshing to see my class take ownership of their learning.

Lina Fidalgo, fourth-grade teacher

Elementary teachers are being asked to think creatively and flexibly about the types of representations that will make mathematics meaningful in their classrooms. The *Principles and Standards for School Mathematics* (NCTM 2000) notes, "a major responsibility of teachers is to create a learning environment in which students' use of multiple representations is encouraged" (139). Its Representation Standard states that instruction should enable all children to:

1. create and use representations to organize, record, and communicate mathematical ideas;
2. select, apply, and translate among mathematical representations to solve problems; and
3. use representations to model and interpret physical, social, and mathematical phenomena. (67)

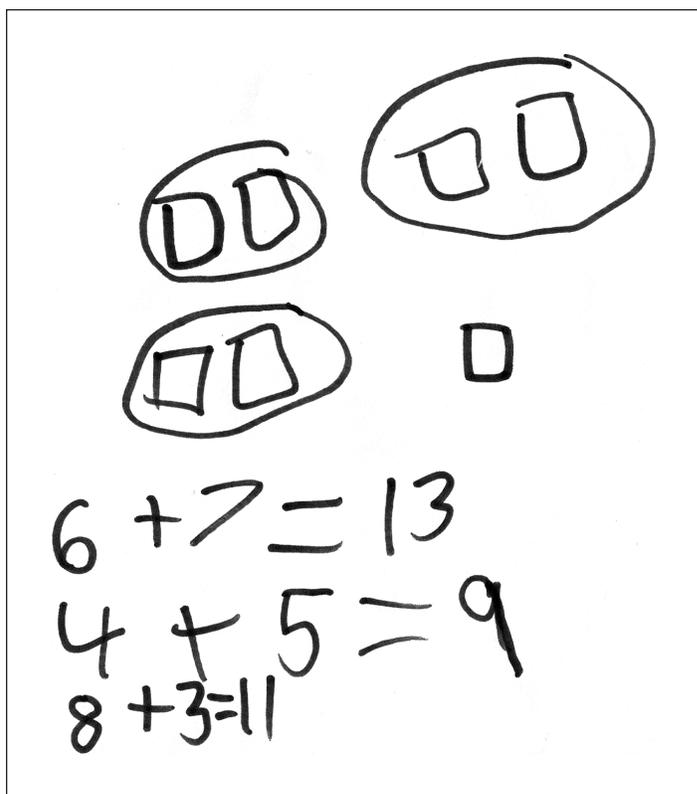


Figure 6-1 Representations for reasoning about even and odd sums

The term *representation* refers to both the *process* of representing an idea and the *product*, or result, of that process (NCTM 2000). For instance, to explore whether the sum of an even number and an odd number is even or odd, children may represent their thinking using number sentences consisting of specific even and odd numbers, or with pictures of squares, paired together, from a collection of squares (see Figure 6-1). Either is a representation of the *process* of reasoning about the sum of an even number and an odd number. Once children determine that the sum will always be odd, they may represent this *product* in words as “the sum of an even number and odd number is always odd,” or they might use the same number sentences or pictures to represent the result—or product—of their thinking.

Representations can take many forms. They may involve words, symbols, pictures, diagrams, tables, or graphs. With functional thinking, such as in the task Counting Dog Eyes, children might represent the relationship in words (“the number of eyes is twice the number of dogs”), symbols ($E = 2D$), tables (see Figure 3-1, page 33), or graphs (see Figure 3-17, page 47). Representations may be given in oral, written, or tactile forms. The choice

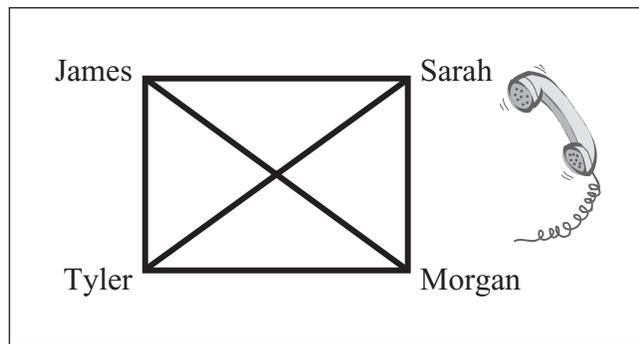


Figure 6-2 Line segment diagram tracking phone calls between four people

of representation varies by both grade level and the mathematical experience of the learner. Some children need access to handheld manipulatives, such as pattern blocks, candies, or paper cutouts. Some need to physically act out a process, such as shaking hands with their friends to count handshakes in a group. Some are content to make drawings on paper, such as diagrams where line segments track phone calls among a group of people (see Figure 6-2).

Typically, the younger the child, the more physical or concrete the representations need to be. For the Outfit Problem (see Figure 6-3; see also Appendix A, page 177, for a solution), first-grade teacher Gail Sowersby created construction paper cutouts of different-colored shirts and pants that students could place on chart paper at the front of the room. Her young students needed to physically see and manipulate the different outfits by pairing the construction-paper cutouts.

Outfit Problem

1. If you have 2 shirts and 3 pairs of pants, how many different outfits* can you make? Show how you got your solution.
2. How many outfits will there be if you have 3 shirts (and 3 pants)? What if you have 4 shirts? Five shirts? Organize your data in a table. Do you see a relationship? Describe it in words or symbols.

Based on your relationship, predict how many outfits there will be if you have 20 shirts (and 3 pairs of pants).

Ms. Sowersby's First-Grade Outfit Problem: If you have 1 shirt and 2 pairs of pants, how many outfits can you make? What if you have 2 shirts? Three shirts?

*Assume an outfit consists of 1 shirt and 1 pair of pants.

Figure 6-3 Outfit Problem

Maggie	Melissa and Sandra	Dana
red shirt, blue pants	red, blue	RS, BP

Figure 6–4 *Representations of outfits*

When children have multiple ways to represent an idea, they can choose representations that are intrinsically meaningful to them. As a result, they are more likely to be successful with the task at hand. Laura Hunt describes how her third-grade students, working in small groups to solve the Outfit Problem, searched for representations that were convincing to them:

Four children, who happened to be distributed among different groups, began to organize their findings after working with the problem for several minutes. For each outfit combination, Maggie wrote the color of the piece of clothing, as well as the piece of clothing's name. Melissa and Sandra simply wrote each different color combination, eliminating any reference to the piece of clothing, while Dana wrote the initial for each color followed by an initial for the piece of clothing it represented (see Figure 6–4).

Many students used an index finger to map out . . . the various outfits that could be made. Students who were using their fingers . . . were having a hard time convincing their tablemates that they had arrived at all the possible combinations.

Some students . . . drew lines to represent the different outfit combinations. Colon actually used different-colored pencils for each of the lines he used to represent the different outfits.

Several students asked if they could color the outfits. It appeared to me that these students had initially looked to others in the group to solve the problem . . . and now they needed proof for themselves.

After about fifteen minutes of small group work, Laura asked children to share at the board the different ways they represented finding all the outfits; the class discussed whether each representation was convincing. From an instructional perspective, several things are significant here:

- Children were encouraged to choose representations that were meaningful to them.

- Children were given time to develop these representations as they explored the Outfit Problem.
- Children were encouraged to share and critique different representations.

These processes support a learning environment where children’s use of multiple representations can thrive.



THINK ABOUT IT 6.1

Think about your classroom practice. What is your role when a task such as the Outfit Problem is being solved? What is the role of your students? Who is responsible for representing the ideas? Are children given time and opportunity to represent their own thinking? Are multiple representations shared and discussed publicly?

As you encourage your students to develop and use multiple representations, help them build connections across these representations so that they can transition from concrete to more abstract ways of thinking. “Seeing similarities in the ways to represent different situations is an important step toward abstraction” (NCTM 2000, 138). For first-grade students, this could mean understanding that a square on chart paper represents an actual shirt (see Figure 6–5). With second graders exploring the Handshake Problem (see Figure 4–2, page 59), this might involve making a connection between actually shaking hands and using the initials of a person’s name recorded on paper to represent the handshake (see Figure 6–6). For third graders tracking the total number of eyes for a varying number of dogs, this could involve converting function tables into graphs (see

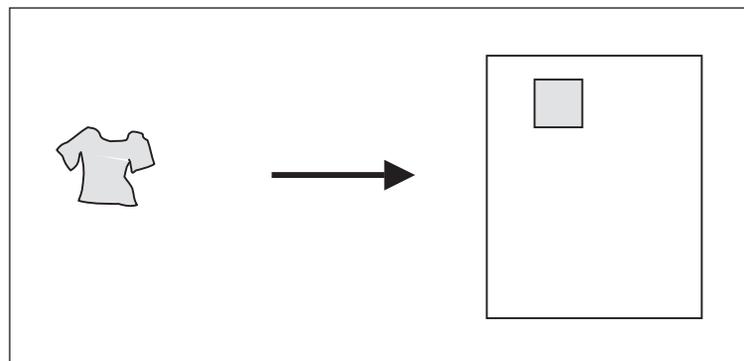


Figure 6–5 *Representations of a shirt*

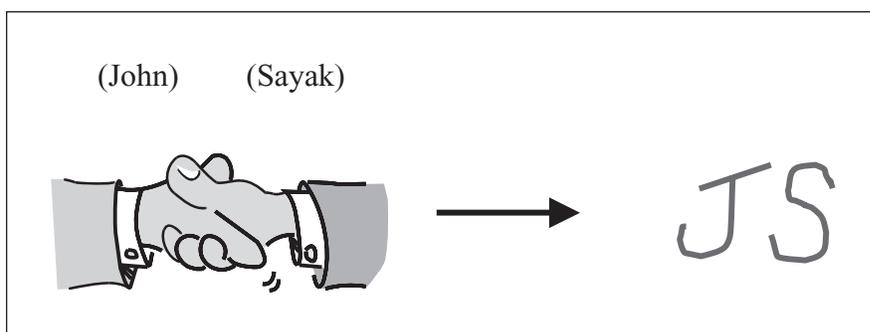


Figure 6-6 Representation of a handshake

Figure 6-7). As children develop flexible ways to think about and represent their ideas, they build a richer, more connected understanding of that idea.

Teach Children to Be Organized and Systematic

Elementary teachers can also help children learn to be systematic in how they represent their thinking. On their own, children often record ideas in random, unorganized ways. Teachers can help children learn to organize their data so that mathematical ideas are closer to the surface. For the Outfit Problem, Christie Demers noticed that her third-grade students had drawn pictures of shirts and pairs of pants and were randomly matching them with no clear strategy for keeping track of what combination was already used. She wrote:

When they kept coming up with different answers each time, I asked them if there might be a better way to keep track of the shirts and pants they

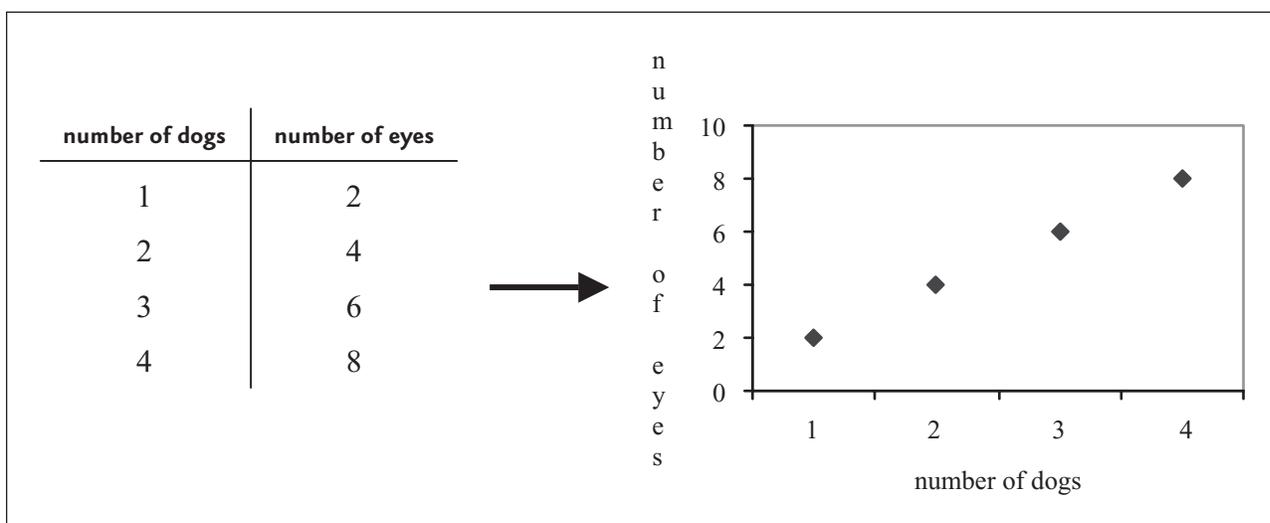


Figure 6-7 Representations of a relationship between the number of dogs and number of dog eyes

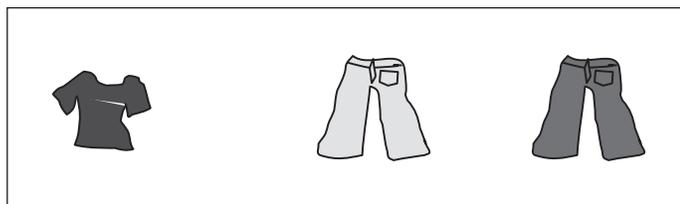


Figure 6-8 One shirt and two pairs of pants

were pairing up. One student answered, “Let’s match up one pair of pants with all of the shirts, then do the same thing for the other two pairs.”

Younger children typically need more explicit scaffolding by the teacher. Cheryl Thadeu, a kindergarten teacher, wrote about the Outfit Problem:

I began by showing children a blue construction paper shirt and two pairs of construction paper pants, one red and one yellow, then asking, “How many different outfits could you make with one shirt and two pairs of pants?” (see Figure 6-8).

After they discussed that an outfit consisted of a shirt and a pair of pants and Cheryl modeled putting an outfit together, she continued:

I made a table on chart paper with one column for shirts and another for pants. I wrote the colors of the shirt and pants that I had used and explained to the children that I was organizing the data this way so we could keep track of the outfits we made and avoid duplicating any outfits (see Figure 6-9).

Then I asked how I could make another *different* outfit using the remaining yellow pants. . . . Nathan responded, “Change the pants,” meaning replace the red pants with the yellow pants. I asked the children, “Is this a different outfit?” All agreed that it was. (See Figure 6-10.)

We counted the number of different combinations we were able to make (two). I then said, “Now we’re going to add another shirt. How

shirt	pants
blue	red

Figure 6-9 Table for recording outfits

shirt	pants
blue	red
blue	yellow

Figure 6–10 *Table for outfits made from one shirt and two pairs of pants*

many outfits could we make with two shirts and two pairs of pants?” (see Figure 6–11).

With help from Cheryl, children were able to make the rest of the outfits quickly and without duplication. She recorded their findings in a table (see Figure 6–12).

Because kindergarten students are only beginning to read and write words and symbols, Cheryl needed to play a more visible role in recording and organizing information in the table. But even at this early age, children can learn to be systematic in how they handle information. The early elementary grades are a critical place for children to learn to use tools and processes that will further their algebraic thinking in later elementary grades. This might involve learning how to organize data in function tables, or what it means to form and test a mathematical conjecture, or how to decompose numbers as a way to think about algebraic properties of number and operation.¹ Thus, while Cheryl’s students did not reach a sophisticated generalization about the number of outfits (they conjectured that the total number of outfits increased each time an article of clothing was added), they were beginning to think about how to be systematic in recording and organizing data, the kind of representations that could be used to do this (e.g., tables), and how to publicly talk about their mathematical ideas.

Summary of Ideas for Supporting Use of Multiple Representations

The key points on page 102 summarize how your instruction can help children use multiple representations to think algebraically.

¹For example, children might decompose 3 as $2 + 1$ or $1 + 2$. This allows them to establish facts such as $1 + 2 = 2 + 1$ (since both expressions are equal to 3) and can lead to an exploration of the Commutative Property of Addition.



Figure 6-11 *Two shirts and two pairs of pants*

- Teach children to be organized and systematic in how they represent their thinking.
- Provide children with grade-appropriate manipulatives that can help them make sense of data.
- When possible, provide tasks that allow children to physically act out a process (such as shaking hands).
- Encourage children to share and explain their representations so that all students learn multiple ways to represent or model a problem.
- Build children’s flexibility with words, symbols, tables, graphs, and other forms of representation. Although one form of representation might be more meaningful than another to a particular student, encourage children to understand and use all types of representations.
- Give children time to explore different representations and their salient features.
- Help children build connections between concrete and more abstract ways of representing their ideas.

shirt	pants	total
blue	red	
blue	yellow	2
blue	yellow	
blue	red	
green	red	
green	yellow	4

Figure 6-12 *Table showing combinations for one shirt and two pairs of pants and two shirts and two pairs of pants*

Question: Ask Questions That Encourage Children to Think Algebraically

One of the most important things you can do to develop children's algebraic thinking is ask good questions. Classroom stories included in this book show that teaching algebraic thinking is often more about questioning than telling. Asking good questions gives *children* the opportunity to organize their thinking and build mathematical ideas. When a teacher tells children what representation to use or how to symbolize a functional relationship or how to justify a particular conjecture, it lessens the chance for children to develop their own thinking.

Not all questions challenge children's thinking. Think about the kinds of questions you ask your students. What do they require of students? Do they call for students to simply recall number facts or perform computations? Or do they call for students to analyze information, to build arguments, and to explain their reasoning? To help you get started, Figure 6–13 offers several categories of questions that can help give tasks more algebraic meaning for children. Although the questions are stated in general terms, you can modify them to fit your particular task and grade level.



TEACHER TASK 6.1

Make a list of the questions you ask during a particular math lesson. What kinds of questions are they? What do they require of students' thinking? How do your questions compare to the questioning strategies described here?

Listen: Listen to and Build on Children's Thinking

Listening is just as important as questioning. In fact, it has been said that teaching is about those who know being silent so that those who don't know can speak. Often, though, this runs counter to what we feel as teachers. It is instinctive for teachers to help, to tell, to explain. However, listening is critical because it helps you understand children's thinking, and you can use this knowledge to guide your instruction. Also, if you are listening, then children are talking and—more likely—actively engaged in their learning.

Whether students are solving an elaborate task or simply reviewing solutions to homework problems, listen to their ideas, strategies, and reasoning and think about how you can extend their algebraic thinking. An episode in June Soares' third-grade class illustrates how listening to children's thinking can lead to simple, spontaneous ways to include algebraic thinking. While

Questioning Strategies to Build Algebraic Thinking

Ask children to share and explain their ideas (i.e., strategies, representations, conjectures, and reasoning):

- Does anyone have a conjecture* to share?
- How did you model the problem?
- How did you represent your thinking?
- Why did you use this particular representation? How did it help you find the solution?
- What strategy did you use?
- How did you get your solution?
- What does the n stand for in your relationship?

Ask children to compare and contrast their ideas (i.e., strategies, representations, conjectures and reasoning):

- Marta, do you agree with Jack? Why?
- Did anybody get a different solution?
- How are your ideas different?
- Is there a better way to organize the information?
- Would you use a different argument to convince your friends than to convince the teacher? Why?

Ask children to find and describe conjectures about patterns and relationships:

- Do you notice anything that always happens?
- Do you notice anything that is always true?
- How would you describe what is going on in general here?
- Can you describe your pattern (relationship) in words?
- Can you describe your pattern (relationship) in symbols?
- How did you arrive at your pattern (relationship)?

Ask children to justify their conjectures:

- How do you know your conjecture will always be true?
- How do you know your solution will always work?
- How would you convince your friends?
- How could you convince your parents?

Ask children to develop more sophisticated ways of expressing their mathematical ideas:

- How could you describe this relationship using symbols (letters) instead of words?
- How can we represent this unknown quantity? How can we represent this varying quantity? Is there a letter or symbol we can use to represent it that might be easier than writing out the name of the quantity in words?

Figure 6–13 *Questioning strategies to build algebraic thinking*

*For more on conjecturing and justifying, see the section “Components of Building a Generalization” in this chapter. You might want to revisit this set of questioning strategies once you’ve finished reading about conjecturing and justifying.

reviewing homework solutions of whole-number addition exercises, June asked if the sum $5 + 7$ was even or odd. When Tony used arithmetic to answer the question, she challenged his thinking: “How did you get that?” She asked. “I added 5 and 7 and then I looked over there and saw that it was even,” Tony explained. (Tony pointed to a list of evens and odds recorded on a chart on the wall. Twelve was on the list of evens.) “What about $45,678 + 85,631$? Odd or even?” June asked. “It’s odd,” Jenna explained. “Why?” June asked. “Because 8 and 1 is even and odd, and even and odd is odd.”

June was not only listening, we could say she was listening algebraically. That is, she was listening for children to use arithmetic strategies so that she could create alternative tasks that required algebraic thinking. At this point, her students did not know an arithmetic procedure for finding the sum of 45,678 and 85,631. Using such large numbers² required children to look at structural features of the numbers to determine if their sum was even or odd. Jenna was able to answer the question without arithmetic. She looked at the last digit in each number, then to get her result, she used the generalizations that even plus odd is always odd and that the sum would be even or odd based on whether the sum of the last digits was even or odd. Reasoning from structural properties without attending to arithmetic is an important way of thinking algebraically.



TEACHER TASK 6.2

Make an audio recording of one of your math lessons. Play the recording to determine the percentage of time you spend listening to students. How much time do you spend questioning students? How much time do you spend telling students information?

Generalize: Help Children Build Generalizations

The central goal of algebraic thinking is to get children to think about, describe, and justify what is going on *in general* with regard to some mathematical situation. That is, we want children to develop a *generalization*, a statement that describes a general mathematical truth about some set of data. The three instructional goals described so far—*represent*, *question*, and *listen*—are all critical components in helping children build their own generalizations.

²We refer to this in Chapter 4 as “using numbers algebraically.” That is, numbers are treated algebraically because children are attending to their structural properties rather than arithmetic properties.

The Ability to Generalize Builds over Time

The level of generalization that children reach within an activity or task will differ depending on the particular lesson being taught and the grade level. Often, complex ideas need extended periods of time—days, weeks, months, even years—to build in children’s thinking. For example, with the Handshake Problem, being able to describe a symbolic relationship between the number of people in a group and the number of handshakes among them can develop over varying amounts of time, based on the age and experience of your students, as children learn to use tools such as function tables to interpret correspondence relationships in data, begin to understand quantities that vary, and develop the mathematical language to symbolize functional relationships.

Don’t be discouraged if you find that one algebraic thinking task can consume an entire lesson (or more). Remember that you are asking your students to think at a much deeper level than is required to learn arithmetic skills and procedures. As you observe children begin to develop, explain, and justify complex mathematical ideas, you will find that the payoffs are tremendous.

Components of Building a Generalization³

We can characterize the process of building a generalization in the following way (see Figure 6–14):

1. Children are given a mathematical situation to *explore*.
2. They develop a *conjecture*, or mathematical statement that is either true or false.

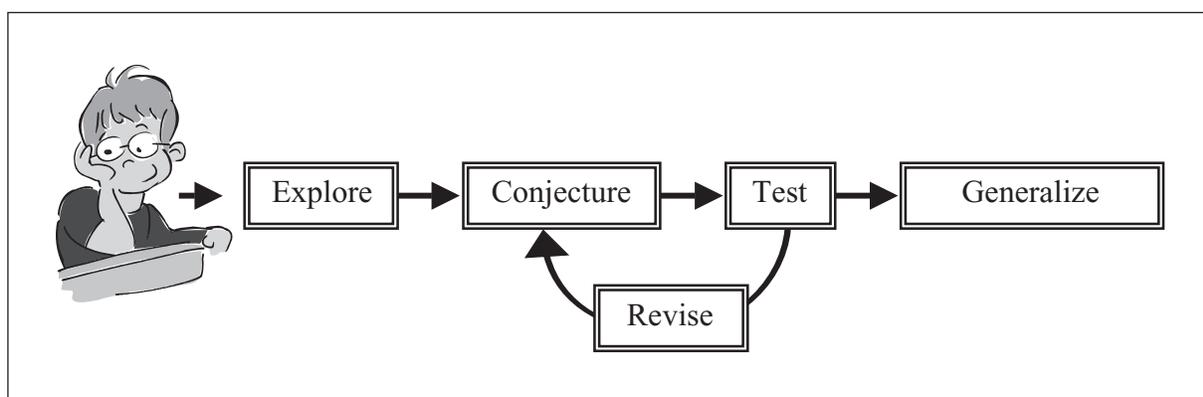


Figure 6–14 Components of building a generalization

³For additional reading on aspects of generalizing, see *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School* (Carpenter, Franke, and Levi 2003). The authors provide a detailed treatment of processes such as conjecturing and justifying in the context of generalized arithmetic.

3. They *test* their conjecture to see if it is true or false.
4. If the conjecture is not true, they can *revise* it and test the new conjecture.
5. If the conjecture is confirmed to be true after sufficient evidence is gathered, it becomes a *generalization*.

Explore Algebraic thinking begins with exploration. Give your students opportunity and time to explore mathematical ideas, both with their peers and working alone. Exploring helps children organize their thoughts and decide how to represent or model their thinking. Whether shaking hands with friends, creating different possible outfits for articles of clothing, solving a family of number sentences, gathering and recording information in a function table, or thinking about turnaround facts, children are thinking mathematically when they explore.

Your role as teacher is to give mathematical—algebraic—purpose to the exploration. This can happen through the type of tasks you choose and the questions you ask. For example, you might begin the Handshake Problem by raising the question, “How many handshakes would there be if everyone in your group shook hands once?” (For the complete Handshake Problem, see Figure 4–2, page 59). The mathematical goal here is to find all possible handshakes. Children might explore this by shaking hands with each other, or drawing pictures to represent the different people in the group. Some students might question what it means to shake someone’s hand: If Nate shakes hands with Mandy, does it count as a different handshake when Mandy shakes hands with Nate? Some will discuss how to represent a handshake, or how to be systematic in tracking handshakes. All of this calls for children to think (at least implicitly) about questions such as What does it mean for two people to shake hands? What method will help me find all the handshakes? How can I represent the different handshakes? and How will I know that I’ve gotten all possible handshakes? The purpose of the teacher’s question (finding all possible handshakes) shapes the direction of children’s thinking. The process of children’s thinking, initiated by the teacher’s question, is what constitutes the exploration.⁴ The point is that the teacher plays a critical role by setting up an issue to explore and allowing children time to explore it.

⁴In contrast, questions in which children need only to recall facts or apply algorithms are not exploratory in and of themselves.

Conjecture Generalizing involves making a *conjecture*. A conjecture is a general mathematical statement that is either true or false on a specified domain. For example, the statement “an even number plus an odd number is always even” is a conjecture that is *not* true, since we can find a counterexample: $2 + 3 = 5$ and 5 is not even. On the other hand, the statement “an even number plus an odd number is always odd” is a conjecture that can be shown to be true for all integers. The statement $E = 2D$, where E is the number of dog eyes and D is the number of dogs, is a conjecture about the total number of dog eyes for an arbitrary number of dogs. A conjecture that is true can be called a generalization.



TEACHER TASK 6.3

Think of at least two different ways you could show that the conjecture “Any even number plus any odd number is odd” is true. Which way do you find most convincing? Why?

When children make mathematical conjectures, they have to organize their thoughts to look for a general relationship. Conjecturing requires them to think carefully about many pieces of information simultaneously and how the pieces are connected. Because conjectures are a critical step in building a generalization, children should see conjecturing as an important mathematical activity. One way to start is through conversations with your students about the meaning of *conjecture* and what it means to make a conjecture. Third-grade teacher Angela Gardiner describes the conversation she had with her students:

I started my lesson by putting these three problems on the board:

$$3 \times 0 = 0$$

$$0 \times 4 = 0$$

$$100 \times 0 = 0$$

I asked my students if these problems were true. They all said yes, confidently. I asked them why the problems were true. The class looked at me a bit puzzled. “What do you mean, why?” asked Simeon. “I mean, why? How do you know that these are all true? Is there a rule that you follow?” I asked. “I know these are all true because anytime

you have zero and multiply it by another number, the answer is zero,”⁵ Simeon explained. I wrote what Simeon said on the board.

After a similar discussion about a second set of number sentences involving multiplying by 1, Angela continued “Nice job, third graders. What you just did was make conjectures,” I told them. “Is a conjecture a rule?” Chandler asked.

“It’s very much like a rule, Chandler. The best way I can explain it is when you are working with math and numbers you sometimes see patterns. You sometimes see that every time you do something, there is a certain result. From the data you collect you can make a rule or conjecture and then you must test that rule to see if it works all the time. If it does, you could say that your conjecture is proven,” I said.

“I know. We do the same thing in science. As scientists, we make *hypotheses* and those are like conjectures, right?” asked Chandler.

“Yes, exactly, Chandler. If you think of it, scientists use the word *hypothesis*, which means an educated guess that you then have to prove is true. In math, mathematicians make conjectures and then they prove those.”

Later in the discussion, Nick asked what makes a number big or small.

“That is a good question, Nick. What do you all think? What makes a number big or small? What rule do you follow?” Angela responded.

“I think a small number is anything below 20,” Patrick said.

“I think that a small number is less than 100,” added Meaghan.

“So, what do we do? Do we make a conjecture about what a small number is?” asked Nick.

“We could, but could we prove that conjecture?” Angela asked.

“No, because everyone has a different opinion of what a small number is,” Callie said.

Through this conversation, Angela helped children see a conjecture as a rule that needed to be tested. It may or may not be true, but it needed to be a statement that could be shown to be true or false. An opinion was not a valid conjecture. She also wrote children’s conjectures on the board. Making children’s conjectures part of a public record by writing them temporarily on a chalkboard or more permanently on charts hung throughout

⁵Children sometimes use their (re)statement of a conjecture as justification that it is true (see Carpenter, Franke, and Levi 2003).

the room shows that they are valuable mathematical ideas. (Typically, those conjectures displayed permanently are true conjectures, or generalizations.)

Also, because this public record helps children remember the generalizations made, they can more easily build on these ideas. That is, they can use generalizations already established to build justifications for other conjectures. For example, once children establish that the sum of any odd number and any even number is always odd or that the sum of any two odd numbers is always even, they can use these generalizations to explore new conjectures. June Soares wrote about her students' thinking: I asked them, "If we added three odd numbers, what would the sum be?" They figured out that the sum would have to be odd because two odds make an even and when you add odd plus even, you get odd.

A conjecture can be expressed in words or symbols. For example, children might express a conjecture about properties of number and operation in words ("anytime you multiply zero by another number, the answer is zero") or symbols ($0 \times a = 0$). They might conjecture a functional relationship between the number of elephants and elephant legs as "the number of elephant legs is 4 times the number of elephants" or " $L = 4E$ " (where L is the number of elephant legs and E is the number of elephants). As discussed earlier, the language children use to express a conjecture will depend in part on their grade level and the richness of their mathematical experiences. The goal is for them to develop more sophisticated mathematical language over time. You will need to adjust your instruction based on your own students and their experiences, but give them the chance to show you how far they can go!



TEACHER TASK 6.4

Introduce the term *conjecture* to your students. Use it in conversations with students so that they become familiar with it.

Test Once children have made a conjecture, they need to test to see if the conjecture is true or false. Not all conjectures are true. Testing involves building a convincing argument, or justification,⁶ that the conjecture is either true or false. Let's think about what this means.

⁶The term *proof* and its derivatives are purposely not used in this book. While you might use this terminology with your students, its meaning can be ambiguous for a diverse audience. What one might view as proof, another might see as only a strong argument and not a proof in a rigorous, mathematical sense.

A conjecture is false if you can produce one case where it fails to be true. This case is called a *counterexample*. As we just saw, the conjecture “any even number plus any odd number is even” is false, since $2 + 3 = 5$ and 5 is not even. That is, we can find a case where an even number plus an odd number does *not* produce an even number. Note that there are many cases where this conjecture fails to be true: $4 + 9 = 13$; $6 + 5 = 11$; $20 + 13 = 33$; and so on. (In fact, it is always false. Why do you think so?) But to show that a conjecture is false, we only need *one* case where it fails. This is because a conjecture can be true if and only if it is true *for all possible cases on some specified domain*. If it fails for one case on that domain, we say the conjecture is false.



TEACHER TASK 6.5

Have a conversation with your students about the number of cases where the conjecture “any even number plus any odd number is even” fails to be true (there are an infinite number of them). Ask students to think about how many different cases are needed to show the conjecture is false. Introduce them to the notion of a counterexample.



THINK ABOUT IT 6.2

Consider the conjecture “an even number divided by an even number is even.” Is it always true? Is it always false? What can you conclude about this? How would you revise the conjecture so that it is always true? (See Appendix C, page 199 for an explanation.)

Give your students experiences with showing a conjecture to be false. False conjectures are easy to construct. One way to do this is to modify a true conjecture so that it no longer holds. For example, simply changing the numbers in a (true) functional relationship will make it a false relationship. A conjecture that the number of dog eyes on D dogs is $3 \times D$ can easily be shown to be false by looking at a particular case: Although children know one dog has 2 eyes, the conjectured function says that one dog has $3 \times 1 = 3$ eyes. It could happen that children conjecture a false relationship to begin with. Use this as an opportunity to explore what it takes to show a conjecture is false.

As another example, suppose that children have established that addition is a commutative operation. That is, the order in which two numbers are added does not matter. We can express the generalization in symbols as $a + b = b + a$, where a and b are (real) numbers. But what if the operation was changed to subtraction? Is the conjecture that $a - b = b - a$, for (real) numbers a and b , true? That is, does the order in which two numbers are subtracted matter? (Note that this introduces negative numbers!) Since we can find a counterexample ($3 - 4 \neq 4 - 3$), we conclude that subtraction is not a commutative operation. In other words, order *does* matter with subtraction. Thus, the conjecture is false and not a generalization.

Sometimes, a conjecture might be true for one set of numbers, but not another. Help children think about the set of numbers for which a conjecture is true and revise their conjectures to reflect this. For example, when children first learn to multiply, they often overgeneralize that “multiplication makes bigger.” But this is not always true. In fact, it’s easy to challenge this claim when introducing multiplication by thinking about multiplying by zero or one. In later elementary grades, children can use fractions between zero and one to test this claim. In other words, not only is it important for children to investigate *if* a conjecture is true, but also *when* it is true. That is, *for what domain or set of numbers* does the conjecture hold?



TEACHER TASK 6.6

Depending on your grade level, have your students think about the set of numbers (for example, natural numbers, integers, fractions) for which the conjecture “multiplication makes bigger” is true and the set of numbers for which it is false. (See Appendix B, page 193 for an explanation.)

Showing that a conjecture is true ultimately requires building a convincing argument that the claim is true for all possible cases over some domain. But the level of sophistication required to make an argument convincing to someone will vary: The argument that convinces a child will likely not convince a mathematician. In fact, a rigorous mathematical argument—one that a mathematician might make—requires certain tools of logic that young children are not expected to know and understand.

But elementary teachers play an important role in helping children understand the importance of justifying their mathematical ideas and building arguments that are increasingly sophisticated mathematically. Carpenter, Franke, and Levi (2003) describe three levels of arguments or justifications that children make: (1) appealing to an authority figure (a conjecture is true

because “the teacher said so”); (2) looking at particular examples or cases; or (3) building generalizable arguments. Let’s examine these levels.

Children’s justifications will often use simple empirical arguments based on testing a number of specific cases (Carpenter, Franke, and Levi 2003; Schifter in press). For example, to show that the sum of two even numbers is even, children might look at examples of two even numbers added together and base their argument on these results. They might argue that, since $2 + 4 = 6$ (even), $2 + 6 = 8$ (even), and $4 + 10 = 14$ (even), then the sum of any two even numbers will be even. While the number of cases they look at will vary, the essence of their argument is based on looking at whether the conjecture holds for particular examples.

But think about how you might guide your students toward more general arguments. To start with, you can encourage them to use different *types* of numbers in an empirical argument, rather than just a random selection of numbers.

In a conversation with her third-grade students about conjectures, Angela Gardiner asked,

“We now know what a conjecture is, but how do we prove a conjecture is true? How do we know that any number times 0 equals 0?”

“We would have to try multiplying a lot of numbers by zero,” Josh answered. “We would have to try different numbers, too,” added Allie.

“Good, but what do you mean by different numbers, Allie?” Angela asked.

“Like some even and some odd,” Allie replied. “Some big and some small, too,” added Joey.

Rather than using a random set of numbers, Allie and Joey wanted to select from different types of numbers. This is an important step toward trying to construct a more general argument because it does not depend on checking a random set of examples. The reasoning is that if a conjecture holds true for a particular type of number (for example, an even number), then it likely holds for all numbers of that type. In this sense, a number is used as a place holder or representative of a class of numbers and not for its specific numerical attributes.



THINK ABOUT IT 6.3

Think about a conjecture your students have made recently during math class. What type of argument did they develop to convince themselves, you, or their peers that their conjecture was true?

But children's arguments can—and should—extend beyond checking particular cases, even with well-chosen numbers. One way to do this is to encourage children to use previously established generalizations as the basis for their arguments. As noted earlier, June Soares' third graders constructed a more general argument when they reasoned that the sum of 3 odd numbers was odd because "2 odds make an even and when you add odd plus even, you get odd." That is, they invoked two previously established generalizations ("the sum of any 2 odds is always even" and "the sum of any odd and any even is always odd") to think about what would happen when any 3 odd numbers were added. They did *not*, however, use an empirical argument of testing sums of 3 particular odd numbers (for example, $1 + 5 + 7$).

In a similar manner, rather than look at examples of sums of 4 odd numbers, Laura Panell's fifth grader Gail reasoned that the sum of any 4 odd numbers would be even "because there are 4 numbers and that is an even amount of odd numbers so it's even" (see Chapter 2, page 17).

When children explore functional relationships, they sometimes test their conjectured relationship empirically by looking at specific cases of the function values. For example, with the Trapezoid Table Problem (see Figure 3-6, page 36), to convince themselves that the number of people who could be seated at t tables was $3t + 2$, June Soares' class "tried many examples from the chart and it worked all of the time. We even tried some big numbers like 100." In other words, once children had developed a function rule, they compared specific function values to the data in their table ("chart") to see if the rule held. So, if the pattern in the function table indicated that 8 people could be seated at 2 tables, they tested their conjectured relationship by substitution: because $3(2) + 2 = 8$ was true, the function was true for that particular example. Once they had tried "many examples," they were convinced that their function rule was accurate.

But in this task, children's reasoning did not end with testing examples. Some were able to argue more generally about the accuracy of their function by reasoning from the context or structure of the problem, using intrinsic features of the task to model the function⁷ (this can be challenging for more complicated functions). By examining the configuration of the tables, children saw that there would always be one person on each end (2 total) and that each trapezoid table always had 3 people on both sides. Because the number of trapezoids varied, then the number of people who could be seated on the sides depended on this varying quantity and could

⁷Chapter 4, for example, illustrates how children might reason from the context of the Growing Snake Problem to develop a functional relationship.

be expressed as “3 times the number of tables,” or $3t$. However, the number of people on the ends was always 2 and was always added to the number of people who could be seated on the sides. June wrote, “They realized the 3 came from the people who could sit “on the top and the bottom” and the 2 “came from the two sides.”

When children’s arguments involve reasoning from the context of a problem, they are connecting intrinsic features of the problem—such as the fact that one person would always be seated at each end of the table configuration—to a mathematical model. This type of justification is not only accessible for children in elementary grades, it also frees their thinking from the limitations of an empirical approach in which they test particular numbers.

With an empirical approach, we only know that the conjecture holds for the particular numbers tested. While this might convince children initially, you can help them begin to appreciate the limitations of this type of justification. With most conjectures, it is impossible to test all possible cases. As a result, there is always a degree of uncertainty about the truth of the conjecture. Help children begin to look for more general arguments by techniques such as reasoning from the problem context or using previously established generalizations. As you develop this sensitivity in children’s thinking, their justifications will become more sophisticated over time.



TEACHER TASK 6.7

Develop a conjecture that you can use to help your students see the limitations of an empirical approach testing specific (numerical) cases or examples. (See Appendix B, page 193 for an explanation.)

While much more could be said about testing conjectures and the types of arguments children build, keep the following points in mind:

- Teach children to always test whether or not their conjectures are true. Give your students experiences with false conjectures. If appropriate, introduce the concept of a counterexample. Remember that a false conjecture can be an opportunity to create a new one!
- Questions such as *Is this always true?* and *Will it always work?* should be a common part of classroom conversations. Help children

think about the domain or set of numbers for which a conjecture is true (or for which it is false).

- Help children think about what it takes to convince someone that a conjecture is true or false. What is a good, convincing argument? Keep in mind that the answer to this will vary based on children's mathematical understanding.
- Help children articulate their arguments clearly and compare them with their peers.
- Help children move beyond empirical arguments that test particular cases or numbers to more general arguments, such as those based on reasoning with previously established generalizations or reasoning from the context of the problem.
- Remember that a generalization is only as strong as the argument on which it is built.

Revise When children encounter a conjecture that is not true, encourage them to revise it to look for a true conjecture. Eventually, children will begin to do this on their own. For example, if they are looking for a functional relationship, they soon learn to revise a function until the relationship holds. Angela Gardiner's account of third graders solving Growing Caterpillar illustrates this (see Figure 6–15).⁸ In this vignette, after Meg used data in the function table to show that Jak's conjectured relationship was false, the class worked to revise the function.

Generalize At this point, most of the work in building a generalization—exploring, conjecturing, testing, and revising—has been done. The final stage is to focus attention on what has been created (a generalization) and on its final form of expression, whether words or symbols. Children should view building a generalization as an important mathematical activity. It is at the heart of algebraic thinking. Show them you value their work—decorate your walls with the generalizations they build!

Building a Generalization: An Example Using Functions

Let's look again at the Trapezoid Table Problem (Figure 3–6, page 36). June described that her students first counted the number of people who could be seated at tables made of 1, 2, 3, and 4 trapezoid blocks and put these data

⁸See Chapter 3, page 52 for a complete account.

“Okay, what do you think the pattern is?” I asked. “I think it is x times 2 plus one,” Jak said. “How many of you agree with Jak?” I questioned. “I don’t know. I have to do a T-chart,” explained Meg. “Well, then let’s do that together on the board,” I said. With the students’ help, we drew the following T-chart on the board:

x	y
1	2
2	5
3	10
4	17

“Now that we have that on the board, I don’t agree with Jak,” said Meg. . . . Because if it was x times 2 plus 1, then x would be 1 and y would be 3. And, it’s not. It’s $x = 1$ and $y = 2$,” she explained. The class struggled with the pattern for a long time. “I see it, I know the formula!” Joe cried out. “Well, what is it?” I prodded. “It’s $x \times x + 1 = y$,” he said.

Figure 6–15 *Children revise a function*

in function tables. Using the function tables, they found a recursive pattern in the number of people (“add 3 to the number of people”) and used this to determine the number of people seated at 12 tables (see Figure 6–16).

What occurred next helped push children’s algebraic thinking further. June wrote:

Explore

I asked the class to look at the chart in another way. I wanted them to look at the relationship between the tables and the number of people. Find the rule. No luck. I then gave them the hint to see if there was a way to multiply and then add some numbers to have it always work. Jon suggested that we try and find a “secret message.”⁹

After a few minutes, believe it or not, Anthony and Allison started to multiply the number of tables with different numbers starting with one.

⁹*Secret message* was the term June’s students coined to denote functions.

number of tables	number of people
1	5
2	8
3	11
4	14
.	.
.	.
.	.
12	38

Figure 6–16 Function table for the Trapezoid Table Problem

Conjecture { The two children arrived at multiplying [the number of tables] by 3 and then they would have to add 2.

Test { We tried many examples from the chart and it worked all of the time. We even tried some big numbers like 100.

Generalize { We then tried to make a “secret message.” Anthony said that the 3 stays the same so use a ‘t’ for table. This is what he came up with:

$$3(t) + 2 = \text{number of people}^{10}$$

Test: Built justification based on problem structure { They realized the 3 came from the people that could sit “on the top and the bottom” and the 2 came from the 2 ends.

¹⁰Notationally, $3(t)$ is the same as $3t$ or $3 \times t$.

During this activity, June's students

- *explored* how the two data sets were related by looking for patterns represented in the function table;
- *conjectured* a relationship and described it in everyday language (“The two children arrived at multiplying [the number of tables] by 3 and then they would have to add 2”). Once they had tested their conjecture, they later described it more symbolically as “ $3(t) + 2 =$ number of people”;
- *tested* this conjecture by looking at specific cases; further justified their conjecture by reasoning about the physical problem and its relationship to the numbers in their function (“They realized the 3 came from the people that could sit ‘on the top and the bottom’ and the 2 came from the 2 ends.”).
- At the end of the task, they had built a *generalization* which they described in symbols as $3(t) + 2 =$ number of people.

“Posing conjectures and trying to justify them is an expected part of students’ mathematical activity” (NCTM 2000, 191).

Conclusion

As you begin to transform your practice, keep in mind that children’s ability to reason algebraically builds over time. It is helpful to revisit ideas— or even tasks—throughout the school year. This allows children time to reflect on complex ideas and organize their thinking over time. Be flexible in your plans. If there’s an opportunity to spontaneously change an arithmetic lesson (or conversation) into an algebraic thinking one, follow this.

Teaching practices that develop children’s algebraic thinking call on a specific set of skills. This chapter described four important practices: (1) help children learn to use a variety of representations, to understand how these representations are connected, and to be systematic and organized when representing their ideas; (2) ask questions that challenge children to explain and compare their ideas, to build and justify conjectures about mathematical relationships and patterns, and express conjectures in increasingly sophisticated ways; (3) listen to children’s thinking and use this to find ways to build more algebraic thinking into your instruction;

and (4) help children build generalizations through exploring, conjecturing, and testing mathematical relationships. Above all, note that these practices are centered on children. The goal is for teaching to focus on *children's* ideas, *their* reasoning, *their* representations, *their* conjectures, *their* arguments, *their* generalizations—in short, *their* algebraic thinking.



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